DESIGN AND ANALYSIS OF A RESISTIVELY HEATED SURGICAL PROBE

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THESIS

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Resistively Heated Surgical Probe

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ABSTRACT

This thesis examines the use of heat for producing clinical lesions in tissue and presents the design and analysis of a resistively heated surgical probe. The probe temperature is accurately maintained and measured by using a Wheatstone bridge. The probe was embedded in an agar-water test medium and the temperature field emanating from the probe was measured using liquid crystal tapes. The experimental results compare within approximately 10% of a two-dimensional numerical solution. A one-dimensional theoretical model is developed and results are presented to show the relative effects of metabolism and blood flow on the temperature field.

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NOTATION

Symbol	Description	Typical Units
α	Tissue thermal diffusivity	cm^2/sec
α'	Temperature coefficient of resistance	ohm/ohm/°C
c	Tissue specific heat	cal/gm/°C
c_b	Blood specific heat	cal/gm/°C
I	Current	amps
k	Tissue thermal conductivity	cal/cm/°C/sec
m_b	Volumetric blood flowrate	$gm/cm^3/sec$
Q	Heating rate	watts
r	Radial location in tissue	cm
r_o	Probe radius	cm
R	Resistance	ohms
S_m	Matabolic heat generation rate	$cal/cm^3/sec$
t	Time	sec
T	Tissue temperature	°C
T_b	Arterial temperature	°C
T_c	Critical temperature	°C
T_o	Initial tissue temperature	°C
T_p	Probe temperature	°C
V	Voltage	volts
ρ	Tissue density	gm/cm^3
ρ'	Resistivity	ohm-cm

Non-dimensional Groups

$$R = \frac{r}{r_o} \qquad \beta = \frac{m_b c_b r_o^2}{k}$$

$$\theta = \frac{T-T_o}{T_p-T_o}$$

$$\tau = \frac{\alpha t}{r_o^2}$$

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Finally, I wish to express my appreciation to my wife Nola for her patience and understanding while I worked on this thesis.

I. INTRODUCTION

A variety of surgical techniques are presently used to produce clinical lesions. In addition to the familiar mechanical or "scalpel" method, surgical procedures employing chemical, ultrasonic, focused x-rays, radiation, and cooling and heating agencies are used to destroy descrete regions of tissue, normally nervous tissue. The choice of methods depends largely on the nature of the clinical problem. Ideally, the surgeon desires a technique that possesses the following characteristics: 1) Safety, 2) Reversibility, 3) Reproducibility, 4) Sharp delimitation, 5) Hemostasis, 6) Flexibility, 7) Simplicity, and 8) Speed of application. At times, it is not possible to meet all of the above surgical ideals. In such cases, the choice of procedures is based largely on an optimization of the ideals, with safety being of paramount importance. Gengler [10] presents a summary of the various methods, complete with the advantages and disadvantages of each.

In this thesis, the use of heat for producing lesions is investigated. In particular, the design of a resistively heated surgical electrode, henceforth referred to as a probe, is presented. Tissue destruction is accomplished by using the probe as a heating element to raise the temperature of a predetermined region of tissue above a critical level, normally taken to be 55°C [2].

The unique feature of this probe is that its surface temperature can be accurately measured and controlled by using a Wheatstone bridge. As a result, the temperature field in the medium surrounding the probe can be analytically predicted by solving the heat equation prior to applying the probe. This is quite significant when one considers that

the volume of tissue destroyed depends solely on the local tissue temperature.

Temperature fields determined experimentally with the probe embedded in a 0.30% agar - 99.7% water test medium compare within an accuracy of 13% of those predicted using the analytical model. A novel, relatively new, temperature sensing device was used in the experimental phase of the study. Liquid Crystals [12], a material that changes color over a known and well defined temperature range, were employed to obtain a two-dimensional, visual display of the transient temperature field which developed around the probe. By using two separate orientations of the liquid crystals, a three-dimensional picture of the temperature field was inferred. In addition to the highly desirable visual aspect of the liquid crystal material, it also produces a minimum disturbance in the test medium.

The effects of blood flow and metabolism on the size of the lesions produced were also investigated using an analytical model.

II. SUMMARY OF HEATING TECHNIQUES

A. DIRECT CURRENT (dc) METHOD

Localized lesions have been produced in tissue by applying a direct current between two electrodes. An active electrode is inserted into a predetermined location in the tissue, and an inactive electrode, or ground, is placed on some other part of the body. The resistivity of the tissue causes electrical energy to be converted into heat, thus increasing the local tissue temperature. When the tissue temperature is raised above 55°C [2], the tissue is destroyed.

Although small and discrete lesions can be produced by the dc method, it has several disadvantages. Direct current follows the path of least resistivity and tends to have preferred paths in tissue, thus causing irregular lesions. Gas bubbles, a result of electrolysis, form around the active electrode and block the flow of current. In addition, dc devices suffer from polarization which can produce transients that stimulate muscle and nervous tissue [23].

Nerve stimulation does have one advantage. The probe placement can be checked by observing the patient's reaction to the stimulating effect of the current. However, other nerves, as well as those in the lesion, could also be stimulated.

Herrero [14] states another disadvantage of the dc method: "Animals with dc lesions usually had extensive scar tissue both around and infiltrated with the tissue".

It should also be noted that the dc method depends on both the thermal and electrical properties of the tissue. The thermal properties are known within an accuracy of approximately 5% [7] and are relatively

constant throughout the tissue. However, the electrical properties are not well known, are anisotropic and tend to vary with location.

One advantage of the dc method is that the electrodes are small; approximately two millimeters in diameter. Therefore the probes can be painlessly inserted and removed, and cause little damage to the surrounding tissue.

In summary, although the dc method can produce localized lesions, the irregularity of the lesions, control difficulties, and possible side effects of the current, greatly diminish its usefulness.

B. RADIO-FREQUENCY (rf) CURRENT METHOD

The radio-frequency technique produces lesions in much the same way as the direct current method. A long, thin, needle-like probe emits low power continuous wave radio-frequency current. An indifferent electrode is placed on some other part of the body and acts as a ground. As current is passed between the electrodes, the resistivity of the tissue causes electrical energy to be converted to tissue internal energy, thus increasing the local tissue temperature. As in the dc method, lesions are a function of the tissue's thermal and electrical properties.

Gengler [10] presents a detailed analysis of the rf technique. In his summary of the method he states: "The radio-frequency method allows controlled and predictable lesions to be produced within safe limits for physiological and clinical applications."

Unlike direct current, rf current has no preferred path in tissue and no stimulating effect. However, low-frequency alternating current does have a stimulating effect in tissue. Therefore, the probe location can be checked by passing low-frequency current between the electrodes and observing the patient's response.

Brodkey and others [3] discuss another way of assuring proper probe location. They found that a temperature rangeof 40-49°C reversibly blocks nervous function. Thus, after positioning the probe, and raising the local tissue temperature to the reversible range, the patient's response can indicate if the probe is positioned correctly.

Like the dc electrode, the radio-frequency probe is small, can be painlessly inserted and removed, and causes little damage to the surrounding tissue.

The rf method does have several disadvantages. The local tissue temperature may exceed 100°C causing boiling, gas formation, carbonization of the tissue, and adherence of coaguluum to the probe [2]. Also, defects in the probe insulation have been known to cause irregular lesions along the probe tract. However, these difficulties can be avoided by carefully controlling the various physical parameters used in the rf technique.

In summary, the radio-frequency method can be used to produce well-circumscribed lesions and the probe location can be checked by two methods. However, as in the dc method, lesions are still a function of both the thermal and electrical properties of the tissue.

C. HIGH RESISTANCE HEATING PROBE METHOD

In this method, a high resistance electrode is inserted into the tissue. Current is passed through the electrode and the resistance of the electrode causes the probe temperature to increase. Heat is conducted away from the probe into the surrounding tissue. A region of destroyed tissue is formed when the local tissue temperature is raised above approximately 55°C [2].

Theoretically, there is less uncertainty in producing lesions purely by heating than there is in the direct current or radio-frequency techniques. The dc method produces both thermal and electrolytic tissue destruction. As the lesion forms, it offers an increased resistance to the passage of current. Thus, the parameters are changing as the lesion forms. The rf method minimizes electrolysis but still depends on the electrical and thermal properties of the tissue. On the other hand, the high resistance probe depends only on the tissue's thermal properties.

Carpenter and Whittier [4] attempted to produce lesions in animals by using a heating electrode. Although their results were poor, they concluded that it was due to the crude instruments employed. They stated: "It would seem profitable to explore this technique with a more finely perfected thermocautery."

Gildenberg [11] conducted studies with a high resistance heating electrode and a rf probe and found that the heating probe produced less variable lesions than the rf probe. In addition, the rf lesions had significantly more lesions along the puncture canal than lesions formed by the heating probe.

As discussed earlier, a temperature range of 40-49°C can reversibly block nervous function. This technique can be used to check the position of the heating electrode.

Since lesions formed using the resistance heating method depend only on the temperature field surrounding the probe, the surface temperature of the probe must be accurately measured and controlled. This has been one of the difficulties of previous resistance probe designs. Watkins [23] used a probe with a thermocouple placed in the tip. He found that the actual temperature of the probe could exceed the recorded temperature by several degrees if the thermocouple was not

positioned carefully. Thermocouples placed in the surrounding tissue to record the temperature can distort the temperature field and give erroneous readings. Gildenberg [11] inserted a thermocouple one millimeter from the heating electrode to record the tissue temperature. He determined that controlling the current to maintain a constant temperature at the thermocouple was no more accurate than maintaining a constant current.

It should also be noted, that since the lesions produced by the heating probe depend only on thermal destruction of the tissue, the lesions are smaller than those produced by the direct current or radio-frequency current method.

In summary, a properly designed high resistance heating probe can produce small, safe and predictable lesions. The uncertainty is less than the radio-frequency or direct current method since tissue destruction depends only on the thermal properties of the tissue.

III. HIGH RESISTANCE HEATING PROBE

A. THEORY

It is well known that the flow of electric current against any resistance is accompanied by a dissipation of electrical energy in the form of heat. The amount of heat generated is expressed by Joule's law:

$$Q = IV = I^2R \qquad (1)$$

Q = heating rate

V = voltage

I = current

R = resistance

The resistance of a conductor at ordinary temperature is a function of the length and the cross-section of the conductor, varying directly with the length and inversely with the cross-section according to the following relationship:

$$R = \rho' \frac{\ell}{A} \qquad (2)$$

R = resistance observed

ℓ = length of sample

A = cross-section area of sample

The proportionality constant ρ' is called the resistivity of the material. Therefore, all other factors remaining equal, the material having the highest resistivity will generate the most heat.

All metals exhibit some change in resistance with temperature. For pure metals, there is a definite and constant relationship between the temperature and the resistance of the metal. This characteristic is used in electrical-resistance thermometers where temperature is

indicated by measuring the change of resistance. Since electrical resistance measurements can be made with high accuracy on bridge circuits, this type of instrument can give precise temperature measurements.

The theory of the resistance thermometer was applied to the design of the high resistance heating probe. From the above discussion, a given metal will have a particular resistance at a particular temperature. The inverse of this was used in designing the probe. That is, imposing a particular resistance on a metal should cause the metal to come to the corresponding temperature for that resistance. The resistance can easily be controlled by regulating the voltage and current. Thus, the temperature of the probe can be measured and controlled accurately without the use of any external measuring devices.

The change in resistance with temperature of a metal is given by the following expression:

$$R_2 = R_1 [1 + \alpha' (T_2 - T_1)] \qquad (3)$$

R_1 = resistance at temperature T_1

R_2 = resistance at temperature T_2.

The constant α', is the temperature coefficient of resistivity and is expressed as ohms per ohm per degree. It should be noted that, in general, α' is a function of temperature.

B. DESIGN

Equations (1), (2) and (3) express several factors that were considered in designing the high resistance probe. From equations (1) and (2) it can be seen that resistivity should be high to generate the most heat. Equation (3) shows that the temperature coefficient of resistivity should also be high so that the change in resistance with

temperature will be large and therefore easier to measure and control. Ideally, α' should be constant over the range of temperatures to be measured.

Several metals were investigated. Their properties are listed in Table I. In addition, the properties of thermistors were also investigated. Although they are small, rugged, can be made in any shape and have a long life, their change in resistance with temperature is extremely nonlinear. Since this makes temperature control difficult, thermistors were not considered further.

TABLE I

Metal	Resistivity (at 20°C)	Coefficient of Resistivity	Constant
Platinum	10 microhm/cm	.0038 ohm/ohm/°C	
Nickel	7.8	.006	
Nichrome	100	.0004	
Copper	1.72	.0038	
Aluminum	2.8	.0039	
Balco	19.9	.0045	25-100°C
Iron	10	.005	

After comparing the properties of the metals in Table I, Balco was chosen. Balco is the commercial name of an alloy, consisting of 70% nickel and 30% iron, that has the properties that are desirable in a heating electrode. Its temperature coefficient of resistivity is large (α' = 19.9 microhm/cm) and nearly constant between 25°C and 100°C [24].

Electrode insulation was an important consideration in the probe design, since a voltage leak in the probe can cause tissue electrolysis. Balco is available with a standard enamel insulation. The enamel is

made from an oil modified expoxy based resin and has a continuous duty operating temperature of 150°C [24].

Various shaped heating probes were considered. However, to facilitate the construction of the probe, a cylindrical shape was chosen. A probe length to diameter ratio (L/D) of ten to one was used to reduce end effects and to ensure good experimental data.

In order to increase the extent of the temperature field in the test medium, thus increasing the accuracy of the experimental data, the probe was made approximately four times larger than an actual surgical probe.

The heating electrode (see Figures 1 and 2) consists of a skeleton of 0.3894 centimeter (cm) diameter glass tubing. Two 0.038 cm copper wires were inserted in the tubing. One wire was passed out the tip and high resistance 0.0068 cm Balco wire was attached. The Balco wire was tightly wound back along the glass tubing for approximately 3.9 cm. It was then led back into the tubing through a small hole and attached to the second copper wire. The holes at the tip and side of the glass tubing were closed with expoxy. The probe was then given a thin coating of General Electric No. 7031 Adhesive and Insultating Varnish thinned with Xylene to ensure against any insulation defects in the Balco wire. The outside diameter of the completed probe was 0.4145 cm.

C. TEMPERATURE CONTROL

Precision resistance measurements can be made by using a Wheatstone bridge. The bridge (see Figure 3) consists of four resistance arms, a voltage source and a detector. When A is to B, as X is to the standard, there will be no potential difference bewteen points d and b. Therefore, whenever the detector reads zero voltage from point d to b, the bridge

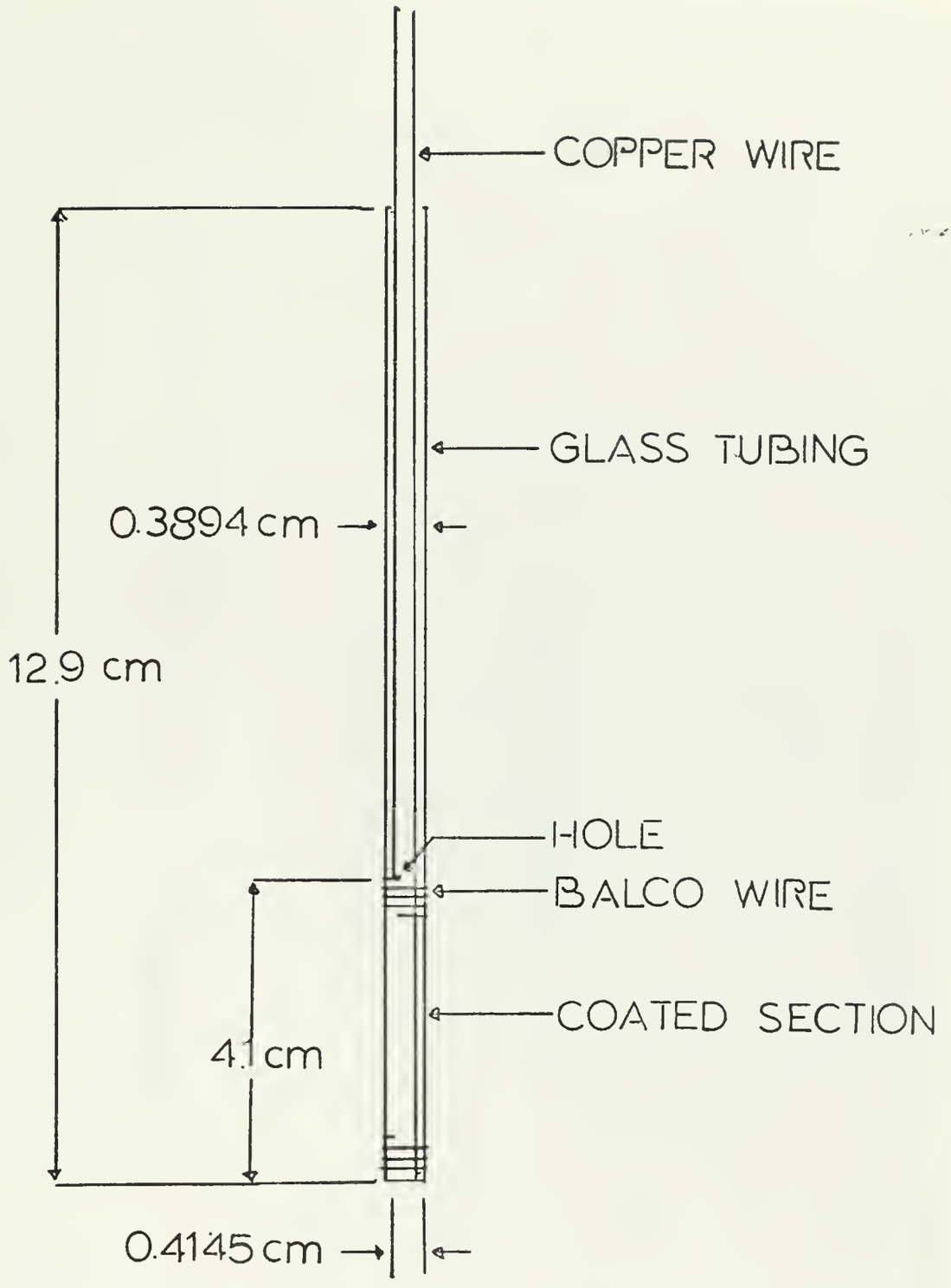

Figure 1. Construction of Probe.

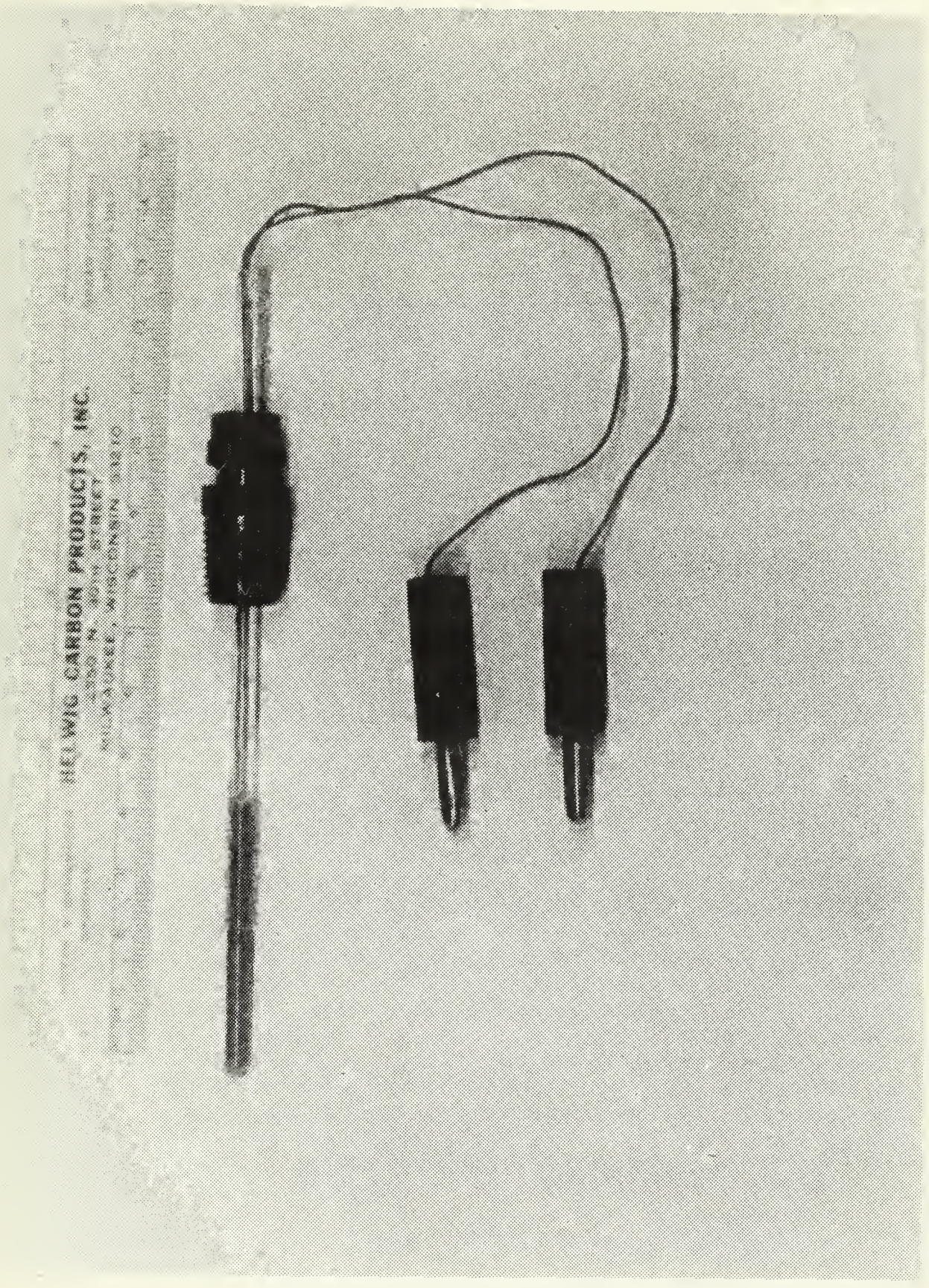

Figure 2. Completed Probe

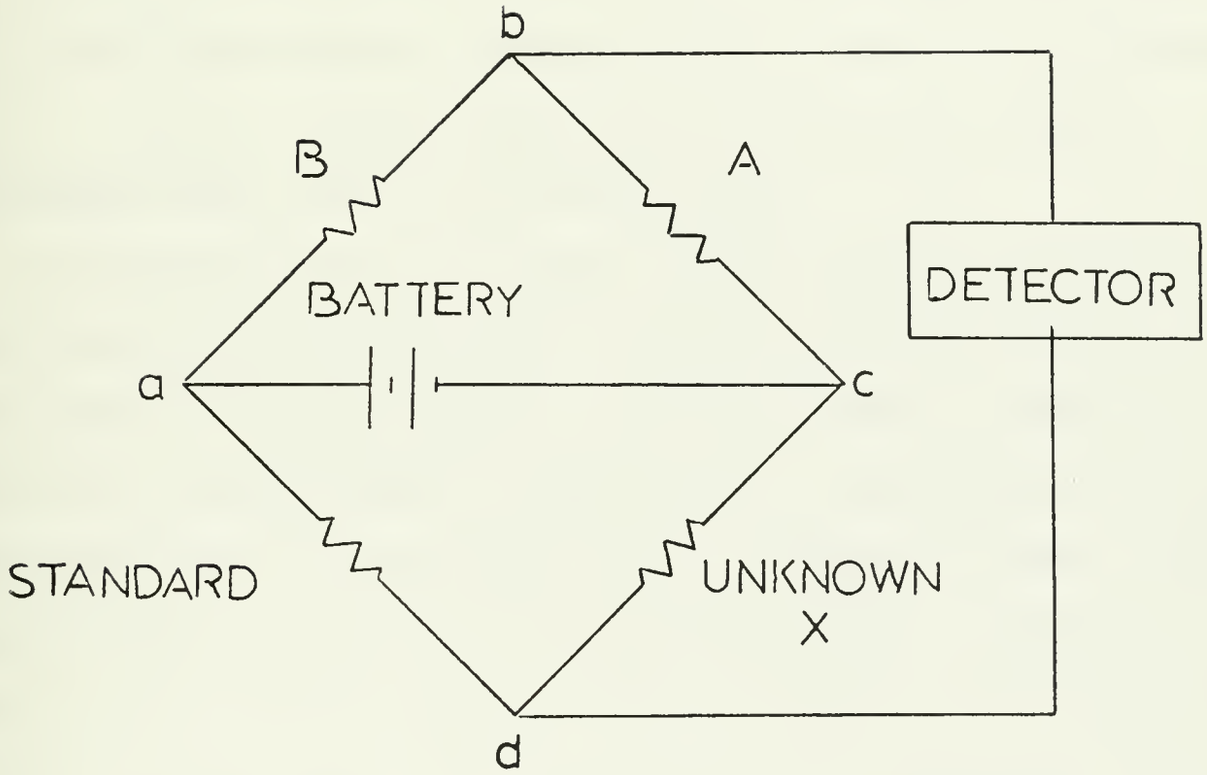

Figure 3. Wheatstone Bridge.

is in balance and the unknown resistance (X) is obtained by using the relationship:

$$X = \left(\frac{A}{B} \right) \times (\text{Standard}) \qquad (4)$$

The probe was calibrated using a Rosemont variable temperature oil bath and commutating bridge. The calibration curve is shown in Figure 5. Because of the experimental apparatus, a probe temperature between 45°C and 50°C was desired. The reason for this choice will be discussed later. It should be noted, however, that in actual use, a probe temperature between 85°C to 95°C would be used.

As discussed earlier, the flow of current against a resistance causes heat to be generated. There is also a change in resistance with temperature. Since the temperature of the probe must be controlled precisely, it was necessary to ensure that the bridge resistances would not change as heat was generated in them. Therefore, 56.2 ohm resistors rated at five watts were used for resistances A and B. Referring to equation (4), it can be seen that since the ratio of A to B is one, the standard resistance will also be the resistance of the unknown (probe).

The standard resistance was wound using Evanohm wire. Evanohm is a commercial wire used in precision resistance standards and is an alloy of 75% nickel, 20% chromium, 2.5% aluminum and 2.5% copper. It has a low temperature coefficient of resistance over a wide range of temperatures [24]. The completed Evanohm standard had a resistance of 356.33 ohms which corresponded to a probe temperature of 48.5°C (see Figure 5).

The final probe circuit (see Figure 4) consisted of the two 56.2 ohm resistors, the Evanohm standard and the probe. A Lambda Model LP 413 FM regulated power supply was used to provide power to the probe. A Weston Model 1240 digital voltmeter was used as the detector.

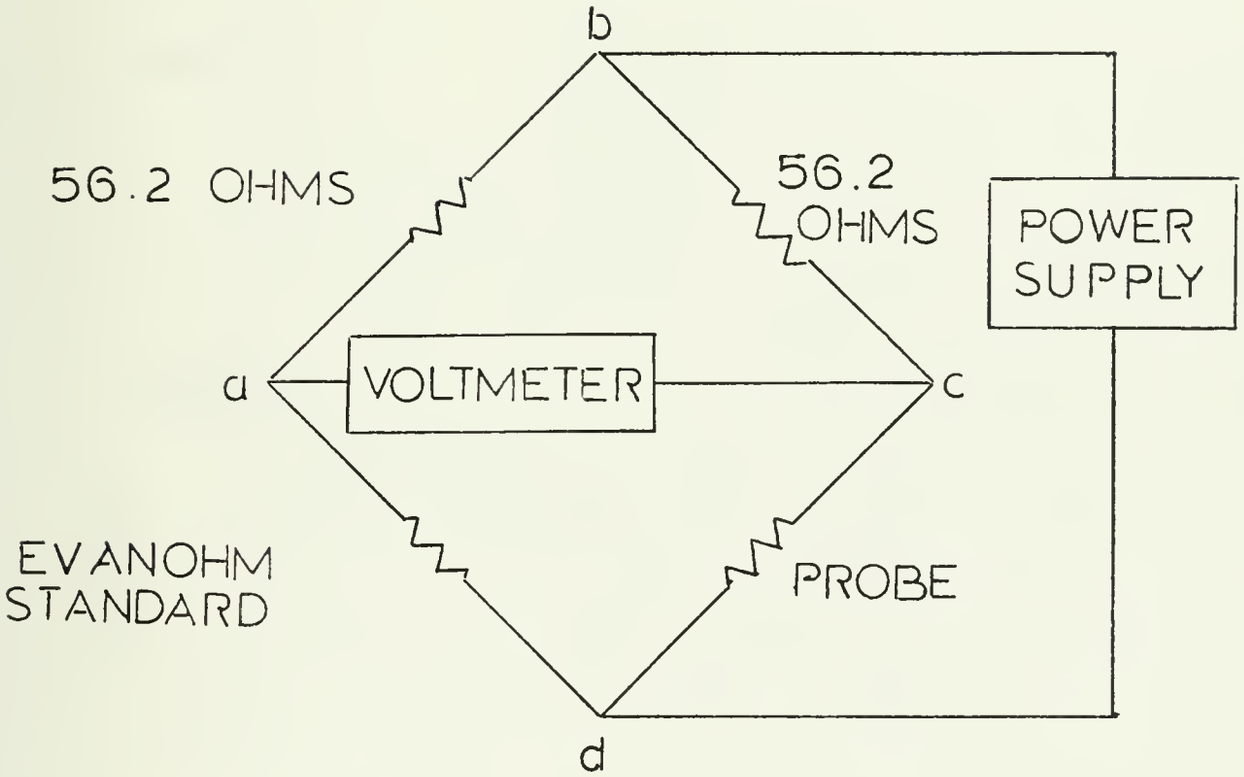

Figure 4. Temperature Control Circuit.

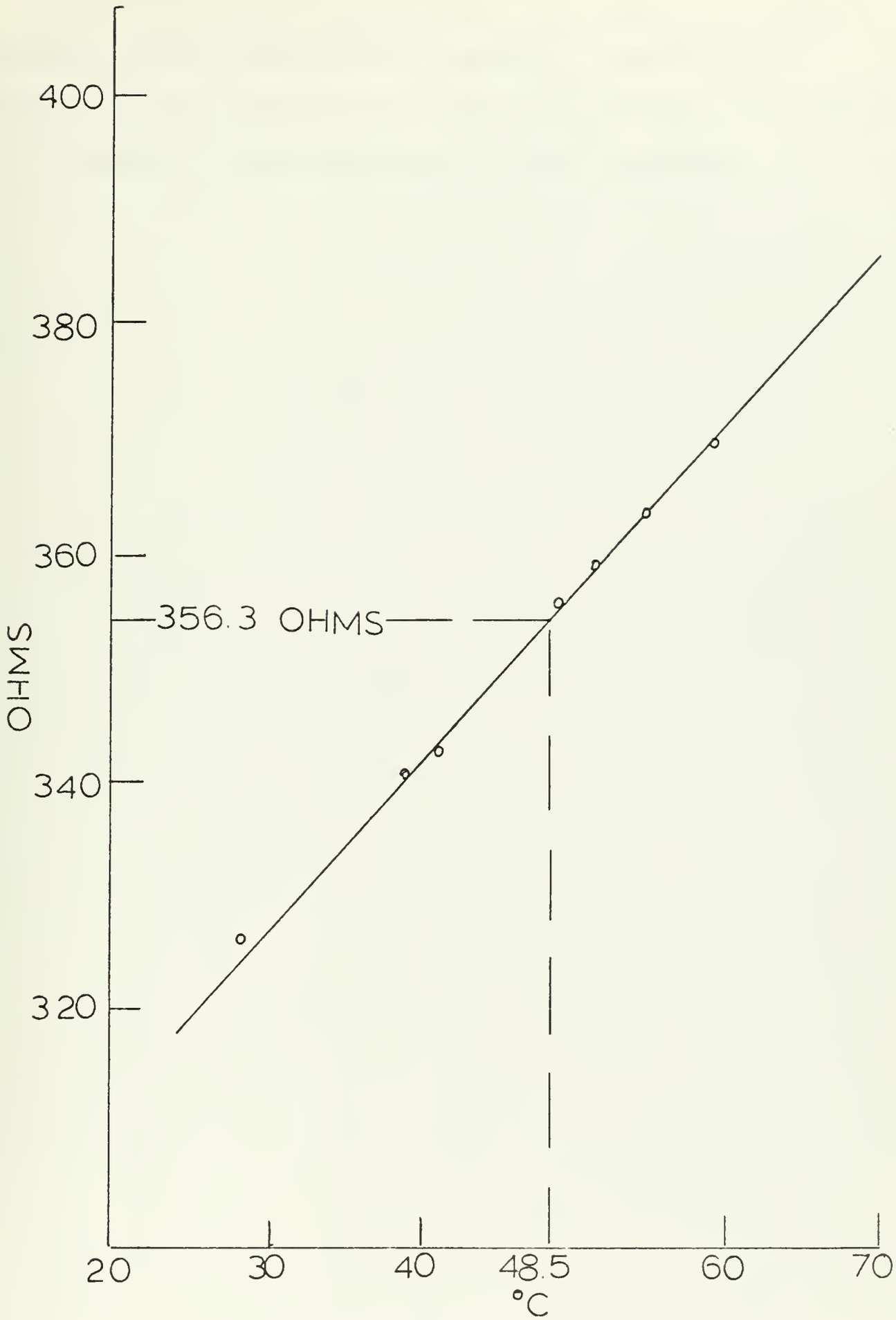

Figure 5. Calibration Curve.

Since the probe had a high temperature coefficient of resistance, the probe resistance was constantly changing as it generated heat. However, by manually regulating the Lambda power supply the bridge could be kept balanced and a probe temperature of 48.5°C maintained.

IV. EXPERIMENTAL APPARATUS

A. TEST MEDIUM

A region of nervous tissue is quite complex. However, to test the heating probe, a simplified medium that had the heat transfer characteristics of tissue was desired. Since the lesions produced by the heating probe are a result of thermal conduction, heat transfer by convection had to be negligible in the test medium. In addition, the medium had to be clear so that the region surrounding the probe could be seen.

Grayson [13] suggested that the water content is a major factor in determining the thermal conduction of animal tissue. Spells [22] investigated thermal conduction in both fluids and tissue and found there was a linear relationship between the conductivity and the water content of the tissue. The linear relationship was particularly close when the water content was above 50%.

Cooper and Trezek [7] obtained results that indicated the thermal properties of human tissue can be predicted within 5% by measuring the water content and assuming the remaining portion of the tissue is composed of equal amounts of fat and protein. They also found that the percentage of water in human tissue varied between 71% and 84%.

The above results indicate that an ideal test medium should have a high percentage of water. Gels would appear to be perfect. Gel consisting of gelatin mixed with an appropriate percentage of water would approximate tissue as a mixture of protein and water, and thermal properties close to those of human tissue could be expected. However, a characteristic of gels is that they melt at approximately 22°C. Since this would lead to convection effects, gel was not used as the test medium.

It was found, however, that agar-water solutions consisting of as little as 0.25% agar are clear, solidify at room temperature, and remain solid up to a temperature of approximately 85°C. In addition, the thermal properties of a 1%, or less, agar-water solution are almost identical with those of pure water, and within 10% of the thermal properties of most human tissues [7]. For these reasons, a mixture of 97.7% water and 0.3% Difco Bacto-Agar was chosen as the experimental test medium.

B. LIQUID CRYSTALS AND TEMPERATURE MEASUREMENT

Under appropriate conditions of temperature and pressure, matter can exist in a solid state, isotropic liquid state or gas state. The majority of organic and inorganic compounds are comprised of a regular, three-dimensional array of molecules or ions in the solid state, i.e., a crystal.

When the crystal lattice is heated, thermal agitation makes it impossible for the molecules to maintain a regular arrangement. At a given temperature, the crystal begins to melt and changes to a liquid. This change, known as the melting point, is usually very sharp and a characteristic of the compund. At still higher temperatures, the liquid boils and passes into the gas phase.

Near the turn of the century it was discovered that some compounds had a well defined melting point but the melt was opaque. The opacity disappeared when the temperature was increased to a definite limit, given a true isotropic liquid. Although this intermediate phase was fluid to some extent, it also exhibited some characteristics of a crystalline solid. This phase was called a mesophase and compounds exhibiting this phenomenon became known as liquid crystals [12].

There are three types of mesophase [12].

(i) smectic - molecules have a parallel orientation and are in layers with their axes at right angles to the layers.

(ii) namatic - molecules have a parallel orientation of their axes.

(iii) cholesteric - molecules have a parallel orientation and are in layers with their axes in the plane of the layer.

The cholesteric mesophase is quite different from the smectic and nematic and possesses unique optical characteristics. Cholesteric liquid crystals having a certain structure exhibit dramatic changes in color with very small changes in temperature. This color-temperature relationship is constant and reappears everytime the liquid crystal passes through a particular temperature range.

Dowden [9] discusses the structure of cholesteric liquid crystals. These crystals are derived from cholesterol or other sterol systems and have a basically planar structure. The axial direction in each successive layer is displaced slightly from that in the previous layer giving a spiral trace. The thickness of each layer is about 3Å and the pitch of the helix is about 5500Å .

The property which is the basis of color behavior is circular dichroism. An incident beam of unpolarized light is split into two components having electrical vectors rotated in opposite directions. One component is transmitted and the other is scattered. The scattered light normally has a wavelength peak that occurs in the visible region of the spectrum.

Temperature changes disturb the intermolecular forces and cause a shift in the molecular structure. This changes the wavelength of the light that is scattered. Thus, cholesteric liquid crystals exhibit dramatic changes in color with very small changes in temperature.

Since the colors scattered by liquid crystals are unique for specific temperatures, the quantitative measurement of temperature is possible to an accuracy of less than 0.1°C [8].

Liquid crystals can be applied directly to a surface. However, a dark uniform layer is normally placed on the surface before the liquid crystals. This provides a standard surface texture, intensifies the resulting colors and isolates the crystals from surface contamination.

Liquid crystals have some application problems. Since they are viscous, applied film thicknesses can easily vary and result in non-reproducible thermal patterns. Direct exposure of the crystals to solvent vapors, oils, dirt and ultra-violet light can cause variations in their sensitivity. However, it is possible to encapsulate the liquid crystal in a tiny capsule (10 to 30 microns), thus shielding it from most of these detrimental effects [17].

Encapsulated crystals are available commercially in a variety of forms that cover a wide range of temperatures. Rochrome liquid crystal tapes manufactured by Hoffman-LaRoche Inc. were used. These crystals were coated on a mylar subtratum with an absorptive black background. The overall thickness of the crystals and mylar was less than 0.01 inches.

The Rochrome crystals are calibrated with an accuracy of 0.1°C and have a response time of less than one second [18]. Parker [19] found that the response time of the encapsulated crystals depends on the crystal thickness. For a thickness of 0.0035 inches, response time was of the order of 200 to 300 milliseconds.

By placing the liquid crystal tape in the test medium, the temperature field around the probe could be seen and temperatures throughout the field accurately measured.

Rochrome liquid crystal tape lot no. AS-70-36 was used in the actual experiments. The color to temperature relationship of this tape was:

29.9°C - red

31.5°C - green

34.2°C - blue

During preliminary experiments with the liquid crystal tapes it was found that the water in the test medium caused the encapsulated crystals to separate from the mylar backing. To prevent this from happening, the tapes were given a thin coating of Varathane and the edges were sealed with a rubber adhesive.

C. TEST CELLS

Two test cells were used to study the three-dimensional temperature field surrounding the probe (see Figure 6). A circular container, five inches in diameter and four inches deep, was constructed of plexiglass and was used to visually observe the radial temperature field. A removable ring was fitted into the cell to support the liquid crystal tape (see Figure 7). The tape was attached to a thin sheet of mylar that was streched across the ring. The probe was inserted through the bottom of the cell and was held in place by a bushing that allowed the height of the probe to be adjusted.

The second cell, a 4 5/8 x 5 1/2 x 6 inch rectangular container, also constructed of plexiglass, was used to observe the axial temperature field near the probe. The probe was inserted through the side of the container. A removable bracket was manufactured to hold the liquid crystal tape lengthwise along the probe (see Figure 7). Because of symmetry, the temperature field on only one side of the probe was measured.

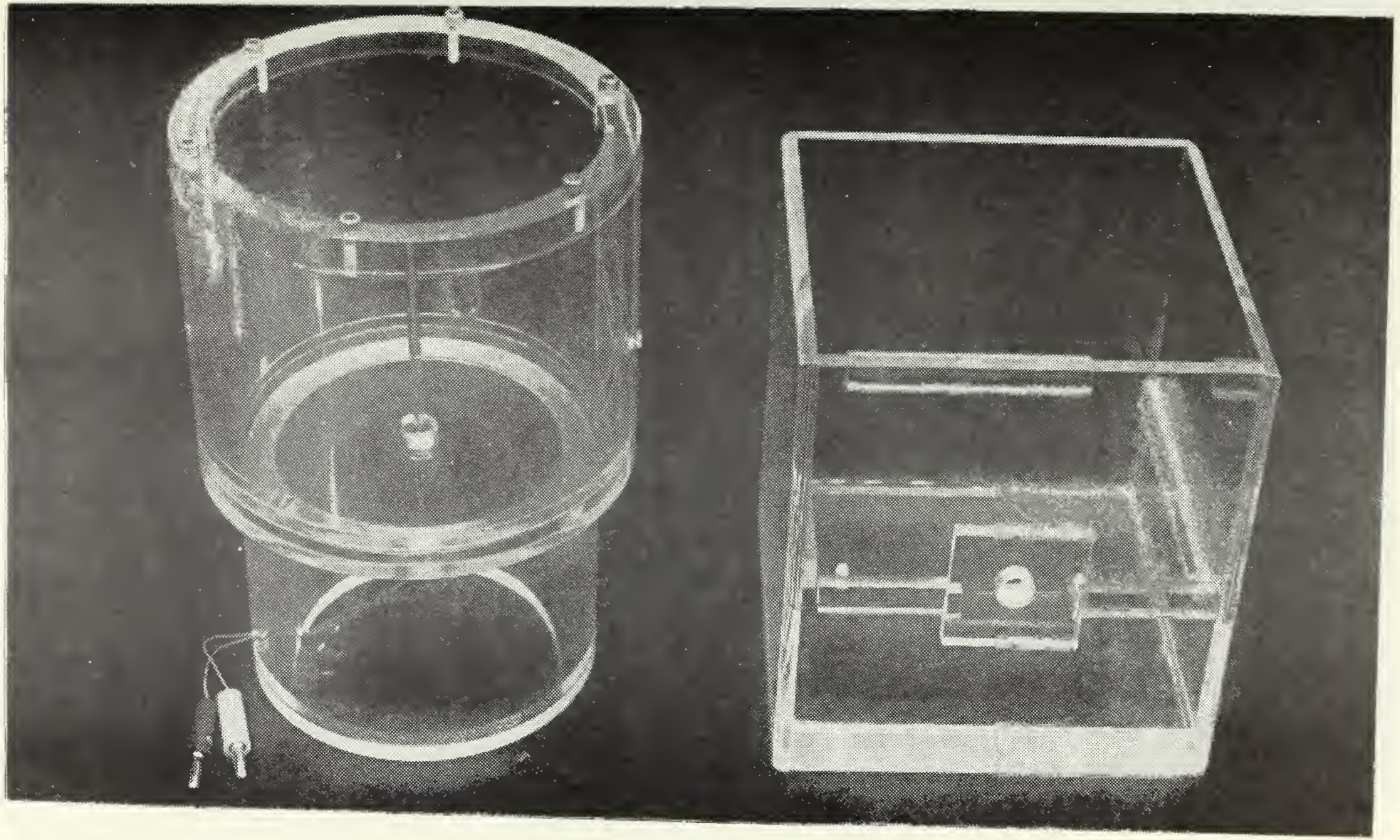

Figure 6. Test Cells

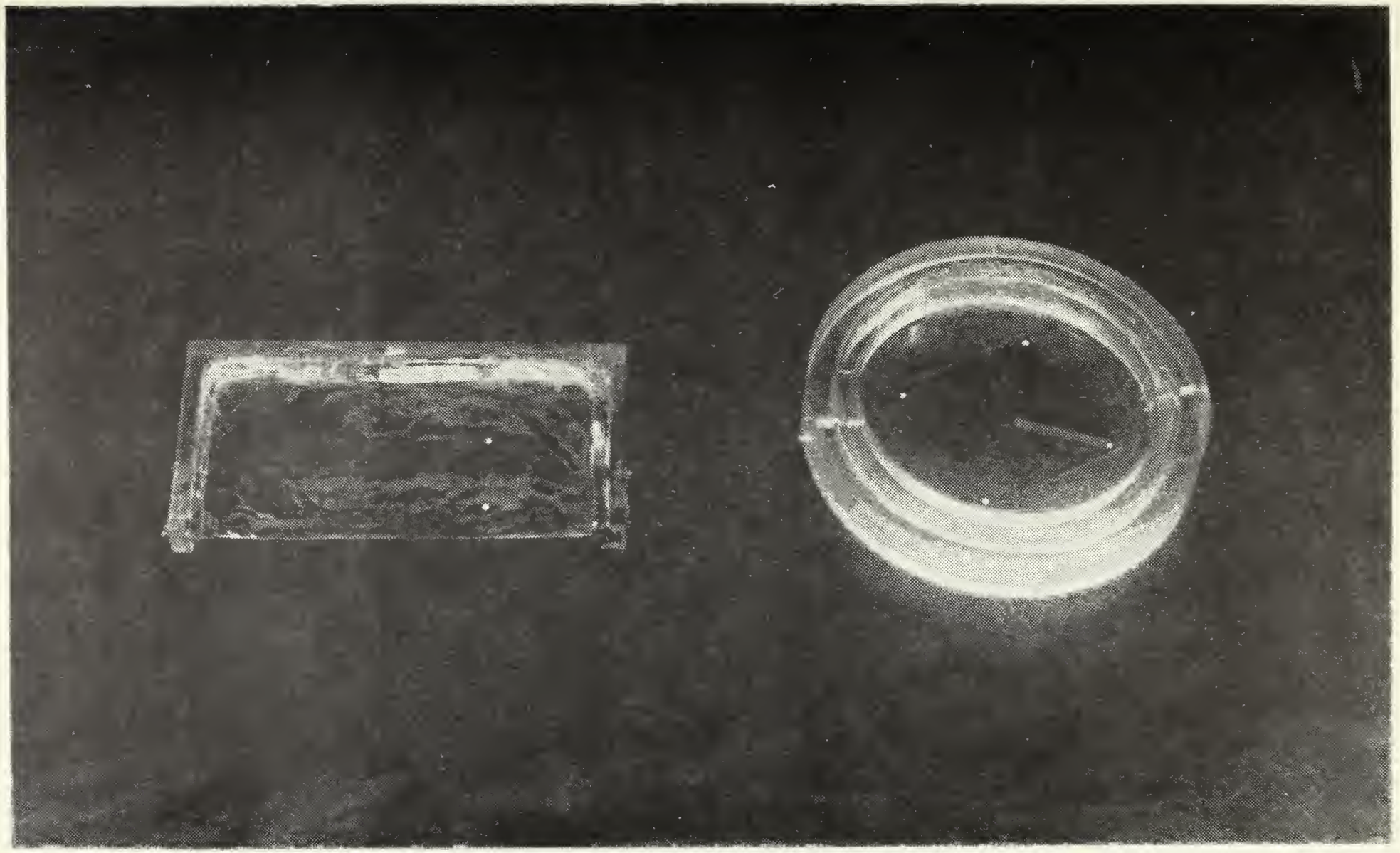

Figure 7. Removable Support Rings for Liquid Crystal Tape.

To ensure that the container walls would not disturb the temperature field around the probe, both containers were constructed so that their dimensions were large compared to the probe dimensions.

V. EXPERIMENTAL PROCEDURE

The test cells were filled with the 0.30% agar-97.79% water test medium for the experimental runs. The probe was positioned in the circular test cell so that there was an equal length of probe above and below the liquid crystal tape. Thus, the temperature field around the midpoint of the probe was measured. The probe was covered by the agar-water medium so that the top of the probe was at least five probe diameters below the surface.

All test runs were conducted at room temperature using Rochrome liquid crystal tape lot no. AS-70-36 to measure the temperature field around the probe. As mentioned earlier, this tape had the following temperature to color relationship:

29.9°C - red

31.5°C - green

34.2°C - blue

A narrow strip of Series 9 Rochrome tape was placed in the test medium to measure the medium's ambient temperature. The temperature to color relationship of the Series 9 tape was:

23.0°C - red

24.0°C - green

27.0°C - blue

The ambient room temperature, normally about 23.5°C, was used as a rough check of the temperature obtained from the Series 9 tape.

Before the containers were filled with the agar-water test medium, four reference marks, 4.57 centimeters apart, were placed on the corners of the tape. Figure 8 is a top view of the circular test

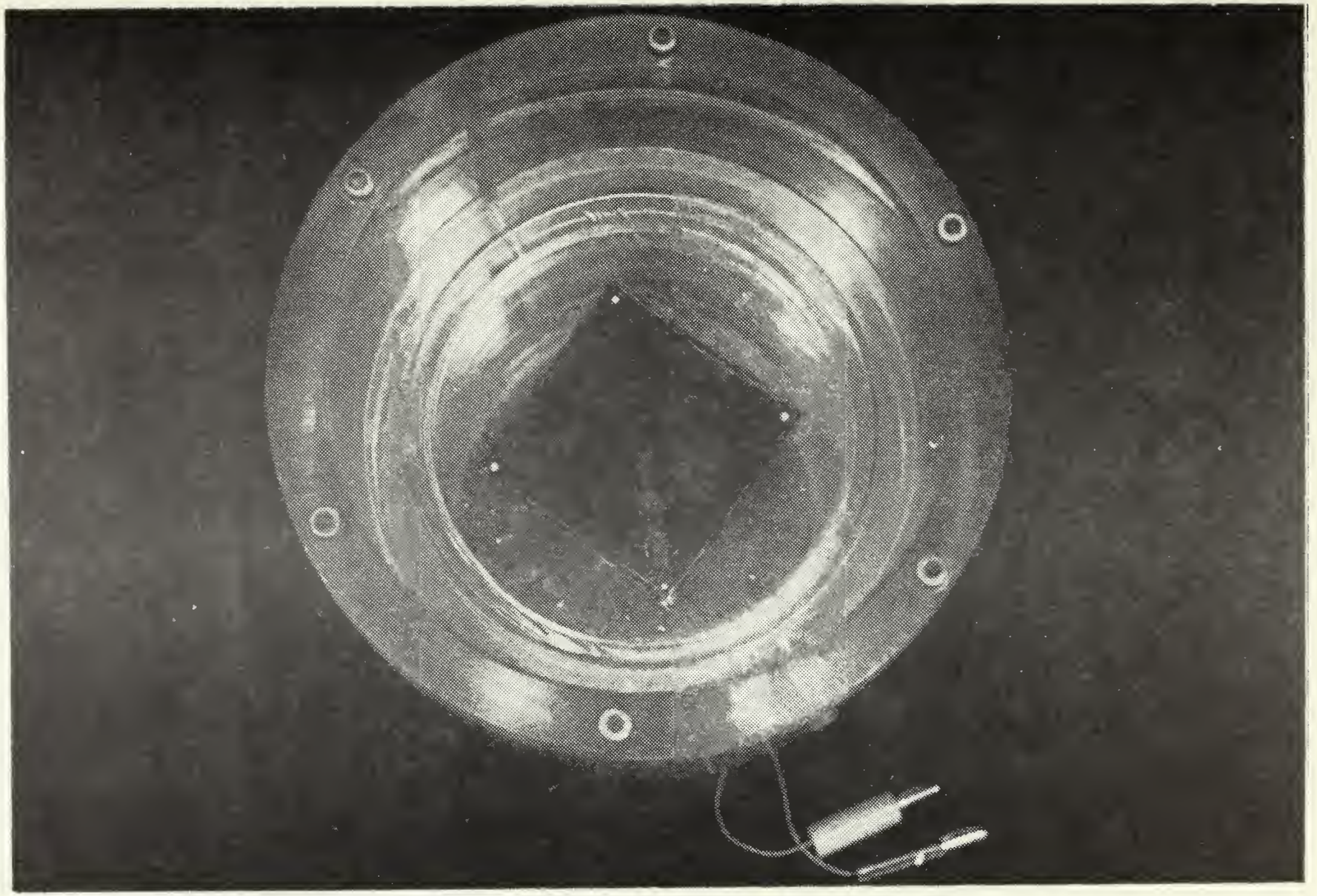

Figure 8. Top View of Circult Test Cell.

cell showing the probe in place, the narrow strip of Series 9 tape, and the four reference marks.

The resistance of the probe was controlled manually by adjusting the Lambda power supply to maintain zero voltage on the detector. Figure 9 is an overall view of the apparatus.

Photographs of the temperature field surrounding the probe were taken using a Graflex 4 x 5 Press camera fitted with a Polaroid Land film holder. Polaroid 4 x 5 Land film Type 57, 3000 speed was used.

Photographs were taken as various increments of time. The outer radius of the temperature field, corresponding to 29.9°C, and the reference marks were measured in each picture. Using the actual distance between the reference marks (4.57 cm) and the probe radius (0.2076 cm), the non-dimensional radius was calculated.

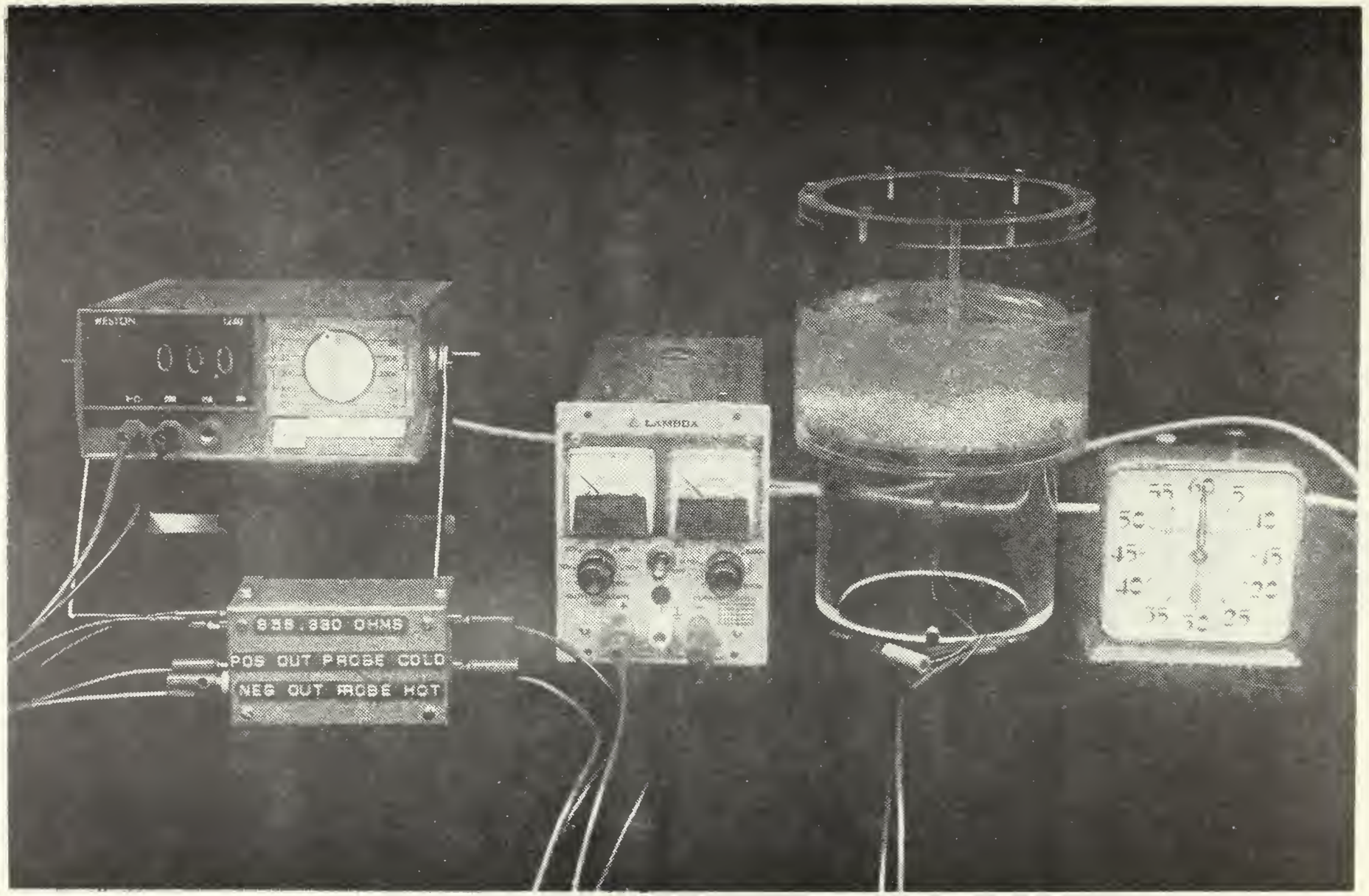

Figure 9. Overall View of Apparatus.

VI. THEORETICAL ANALYSIS

A. ONE-DIMENSIONAL MODEL

1. Metabolic and Blood Flow Effects Neglected

In order to simulate the heat transfer process in the region surrounding the probe, a one-dimensional heat transfer model must be developed. In this model it will be assumed that the tissue is an infinite homogenous medium with known conductivity, density and specific heat and that it is initially at a uniform temperature of T_o. The probe is assumed to be a cylinder of infinite length and radius r_o with a constant temperature of T_p (see Figure 10). Thus, the problem is to solve for the temperature distribution in a region bounded by a circular cylinder.

The general conduction equation is:

$$\frac{k}{r} \frac{\partial \left(r \frac{\partial T}{\partial r} \right)}{\partial r} = \rho c \frac{\partial T}{\partial t} \qquad (5)$$

k = thermal conductivity

ρ = density

c = specific heat

$\alpha = \frac{k}{\rho c}$ = thermal diffusivity

It is now convenient to introduce the following nondimensional variables:

$$R = \frac{r}{r_o} \text{ (distance)}$$

$$\tau = \frac{\alpha t}{r_o^2} \text{ (time)}$$

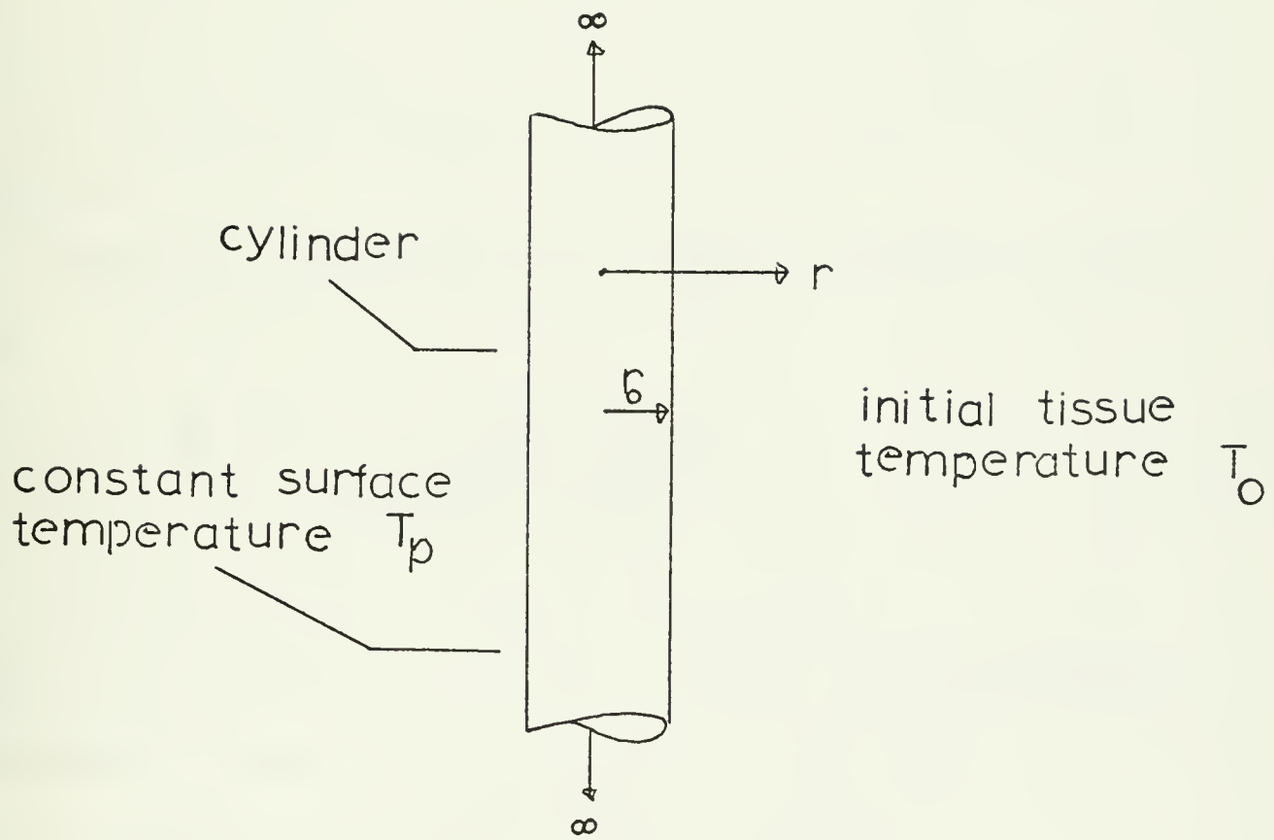

Figure 10. Mathematical Model of Probe.

$$\theta = \frac{T - T_o}{T_p - T_o} \quad \text{(temperature)}$$

Substituting these variables into equation (5) yields,

$$\frac{1}{R} \frac{\partial\left(R \frac{\partial\theta}{\partial R}\right)}{\partial R} = \frac{\partial\theta}{\partial\tau} \tag{6}$$

The boundary conditions are:

(a) at the probe surface ($r = a$, or $R = 1$), $T = T_p$ and $\theta = 1$.

(b) at large distances from the probe the temperature of the tissue remains constant; that is, as $r \to \infty$, $T \to T_o$ or nondimensionally, as $R \to \infty$, $\theta \to o$.

The initial condition is at time $t = o$, $T = T_o$ or $\theta = 0$ for all locations.

Carslaw and Jaeger [5] give the closed form solution of equation (6):

$$\theta = 1 - \frac{2}{\pi} \int_o^\infty \frac{e^{-\tau u^2} C_o(u, Ru)\, du}{u\left[J_o^2(u) + Y_o^2(u)\right]} \tag{7}$$

$$C_o(u, Ru) = J_o(u)\, Y_o(Ru) - Y_o(u)\, J_o(Ru)$$

where, J_o and Y_o are Bessel functions. Equation (7) was evaluated numerically by Jaeger for various values of τ and R [15].

2. Metabolism and Blood Flow Effects Included

Pennes [20] was the first to account for metabolic and blood flow effects in the heat equation. Pennes suggests that the equation take the form:

$$\frac{k}{r} \frac{\partial\left(r \frac{\partial T}{\partial r}\right)}{\partial r} + m_b c_b (T_b - T) + S_m = \rho c \frac{\partial T}{\partial t} \tag{8}$$

where:

m_b = blood flow rate per unit volume

S_m = metabolic heat rate per unit volume

T_b = blood temperature

In this model, the effect of blood flow is treated as a perfect heat exchanger. That is, it is assumed that the blood enters a unit volume of tissue at the arterial temperature, T_b, and acquires the tissue temperature as it exits the volume.

If the metabolic heat generation rate is assumed to be constant, it can be combined with the blood flow term as:

$$m_b \, c_b \, (T_o - T)$$

where

$$T_o = T_b + \frac{S_m}{m_b \, c_b}$$

This corresponds to redefining the reference temperature. Cooper and Trezek [6] show that for brain tissue, T_o is on the order of 0.5°C higher than the arterial blood temperature, T_b.

Equation (8) may be normalized by defining a nondimensional "blood flow parameter":

$$\beta = \frac{m_b \, c_b \, r_o^2}{k} \quad \text{(blood flow term)}$$

Substituting this variable along with R, τ and θ into equation (8) yields:

$$\frac{1}{R} \frac{\partial\left(R \frac{\partial \theta}{\partial \tau}\right)}{\partial R} - \beta \, \theta = \frac{\partial \theta}{\partial \tau} \qquad (9)$$

The boundary conditions and initial condition are the same as were used for equation (6).

Equation (9) was solved with the use of Laplace Transformations and the following closed form solution was obtained:

$$\theta = \frac{K_o\left(\sqrt{\beta R}\right)}{K_o\left(\sqrt{\beta}\right)} - \frac{2}{\pi} e^{-\beta\tau} \int_o^\infty \frac{e^{-\tau u^2} C_o(u, Ru)\, du}{[u+\beta/u]\,[J_o^2(u) + Y_o^2(u)]} \qquad (10)$$

$$C_o(u, Ru) = J_o(u)\, Y_o(uR) - J_o(uR)\, Y_o(u)$$

Equation (10) was solved numerically for various values of R and τ and for values of β = 0.0, 0.1, 0.5 and 1.0 and the results are tabulated in Tables II - XI. The reader is referred to Appendix A for details.

B. TWO-DIMENSIONAL MODEL

The one-dimensional model assumed that the probe was infinite in length. However, the probe consists of a constant temperature cylinder of finite length and a cylindrical insulated stem. The one-dimensional solution does not take end effects into account. Thus, it was desired to have a two-dimensional solution that would include these effects.

A digital computer program, by the code name of Trump [16], was used to determine the two-dimensional transient temperature field around the probe.

Since TRUMP obtains a solution in dimensional form, a nodal netword (see Figure 11) had to be constructed for the geometric configuration of the probe. Due to symmetry, it was necessary to generate the field for only one half of the cross-sectional area. The dimensions of the network were chosen large enough so that the boundary conditions at R = 30 and X = 24.0 r_o did not affect the temperature field during the transient period which was investigated. The network therefore simulated an "infinite" field.

The TRUMP solution gave values of θ at each node point for various values of nondimensional time. The results are given in Tables XII - XVI.

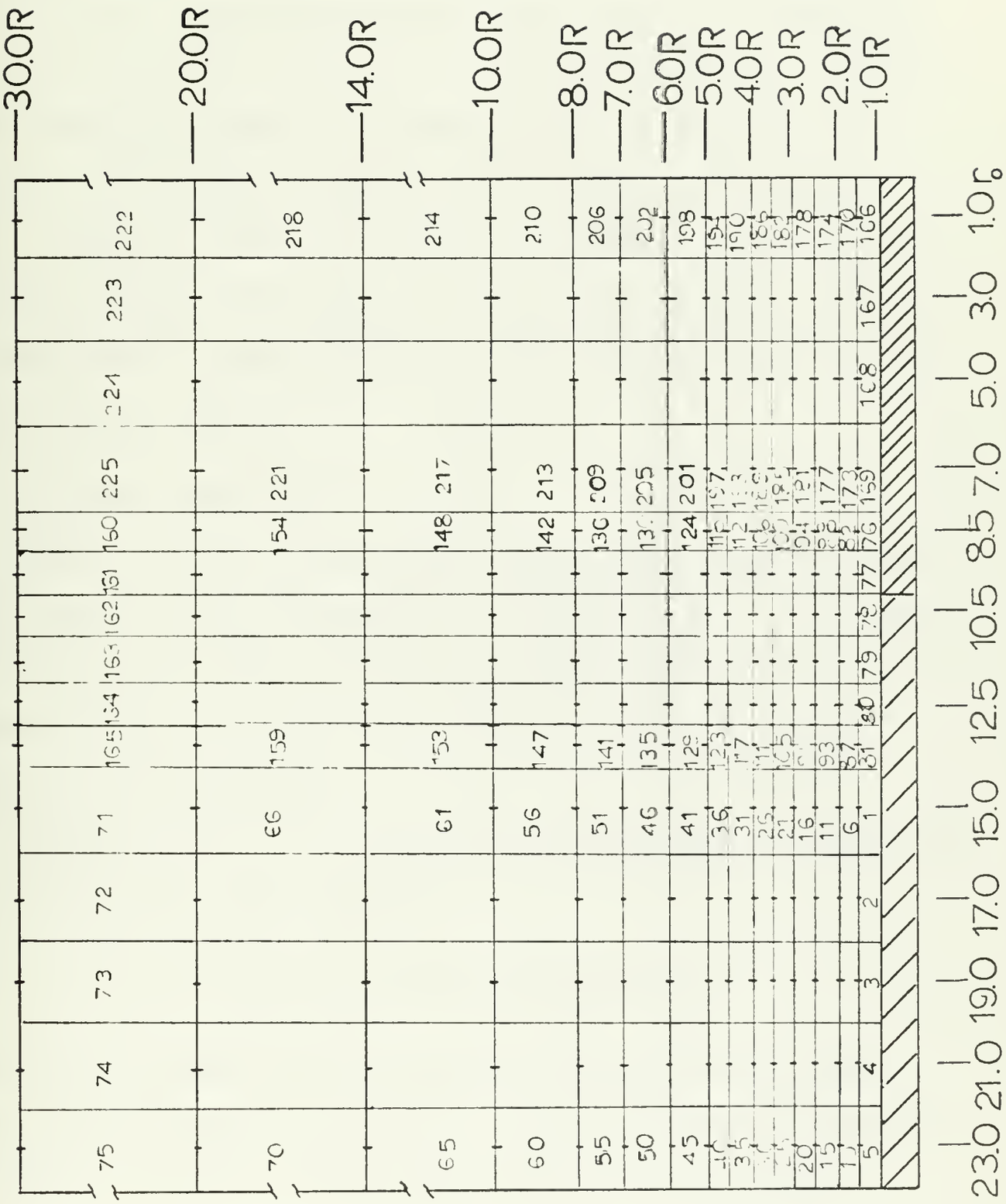

Figure 11. TRUMP Nodal Network.

VII. RESULTS

The preceding section gave details of the three models that were used; a one and two-dimensional model without blood flow and metabolic effects and a one-dimensional model including these effects. Numerical results for the one-dimensional models are given in Tables II-XI. Tables XII - XVI give the numerical results for the two-dimensional model. Values for θ are given at various nondimensional locations (R) and times (τ). The experimental results, differences between the one and two-dimensional models, and the relative effects of blood flow and time will now be discussed.

A. EXPERIMENTAL RESULTS

Figures 12, 13, 14, 15 and 16 are representative pictures of the radial temperature field surrounding the probe taken at τ = 0.0, 1.0, 3.0, 5.0 and 10.0. Figure 17, 18, 19, 20 and 21 depict the axial temperature field at τ = 0.0, 1.0, 3.0, 5.0 and 10.0. The latter pictures also show end effects.

It was convenient to use the following nondimensional variables in calculating the experimental results:

$$\theta_c = \frac{T_c - T_o}{T_p - T_o} \quad \text{(critical temperature)}$$

where, T_c is the temperature corresponding to a given liquid crystal color. In the experimental runs, this color was red and corresponded to a temperature of 29.9°C. In a surgical application, T_c is the temperature at which tissue destruction begins.

$$R_c = \frac{r_c}{r_o} \quad \text{(radius of lesion)}$$

The experimental data and results are given in Table XVII.

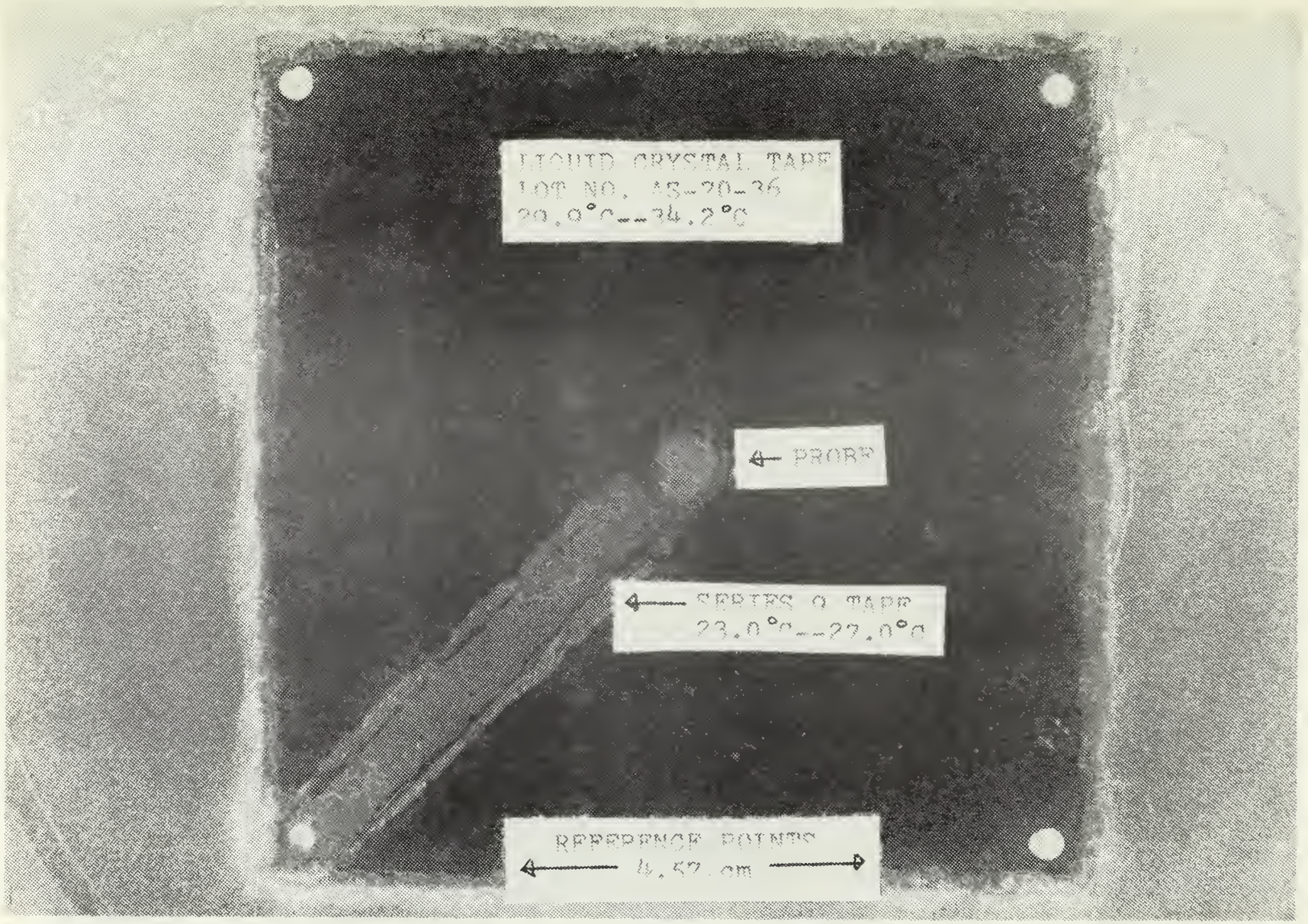

Figure 12. Radial Temperature Field for τ = 0.0.

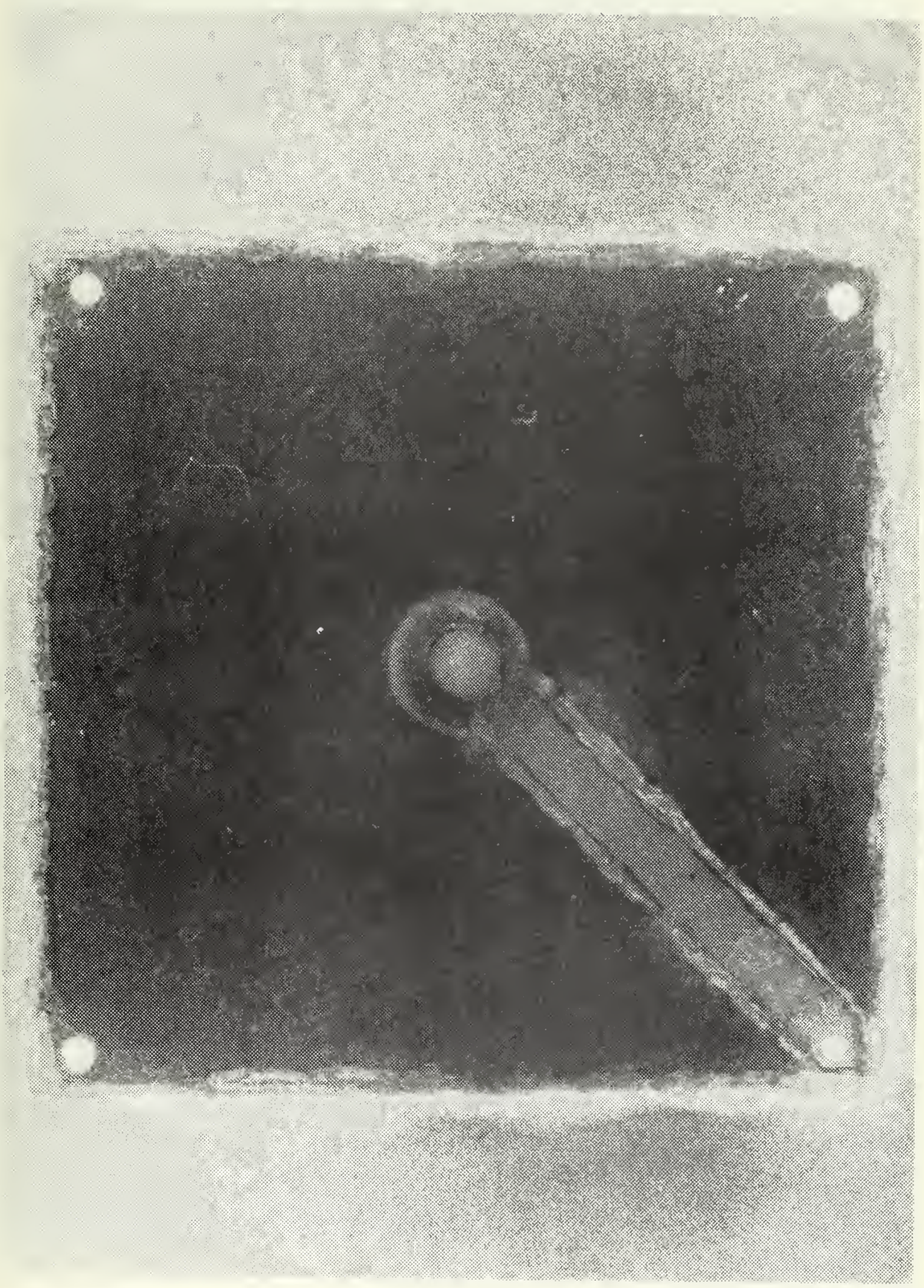

Figure 13. Radial Temperature Field for $\tau = 1.0$

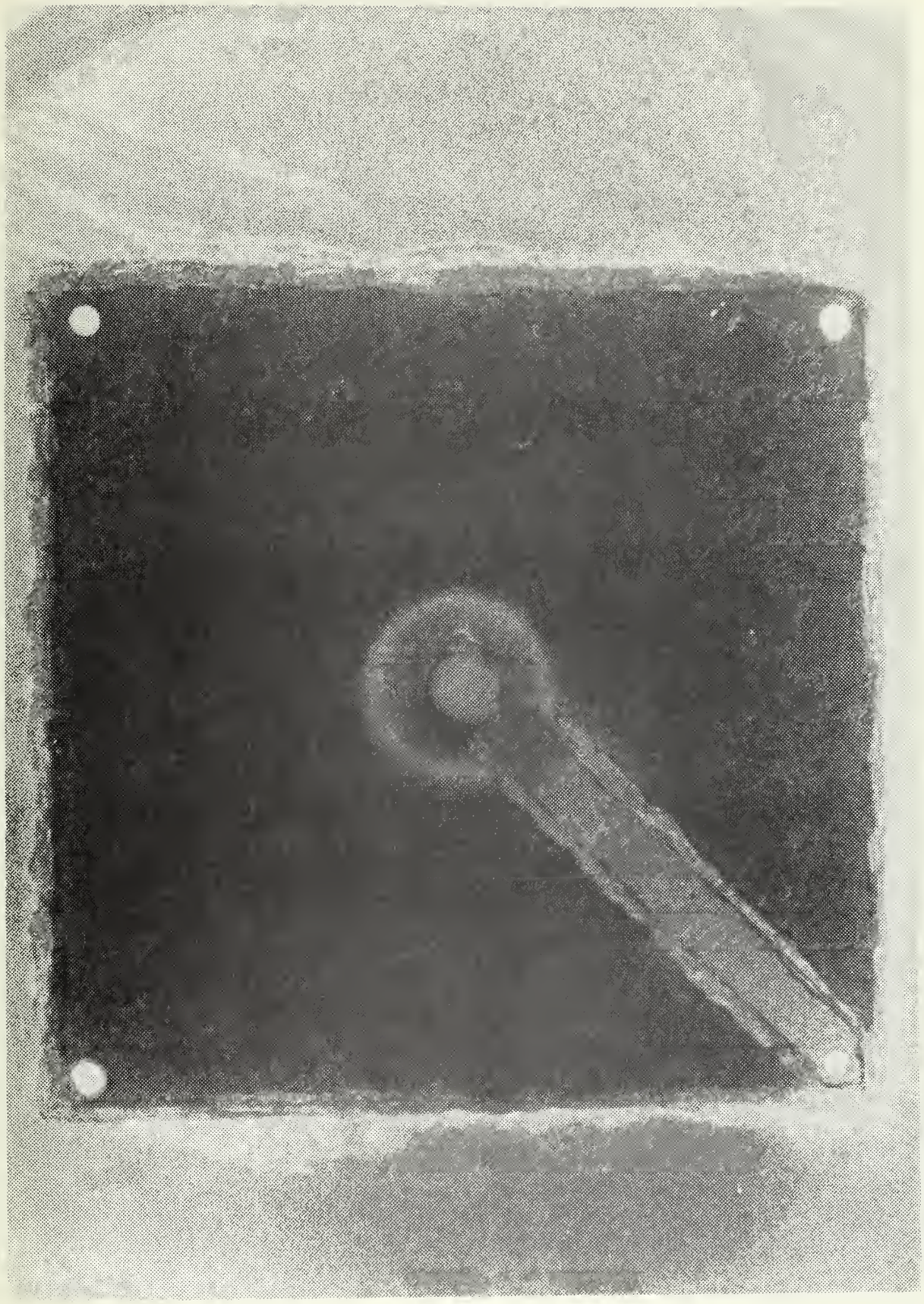

Figure 14. Radial Temperature Field for τ = 3.0.

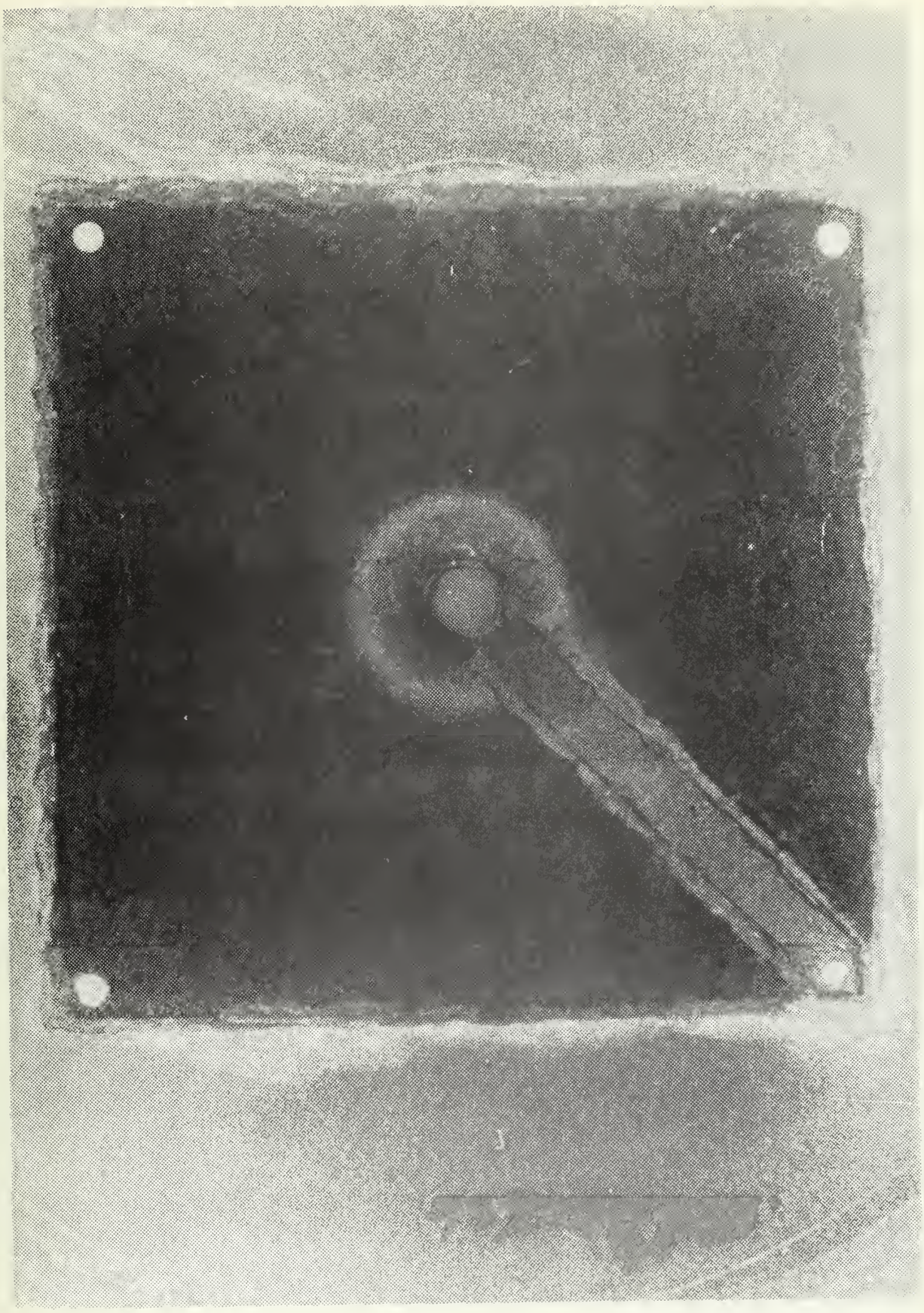

Figure 15. Radial Temperature Field for $\tau = 5.0$.

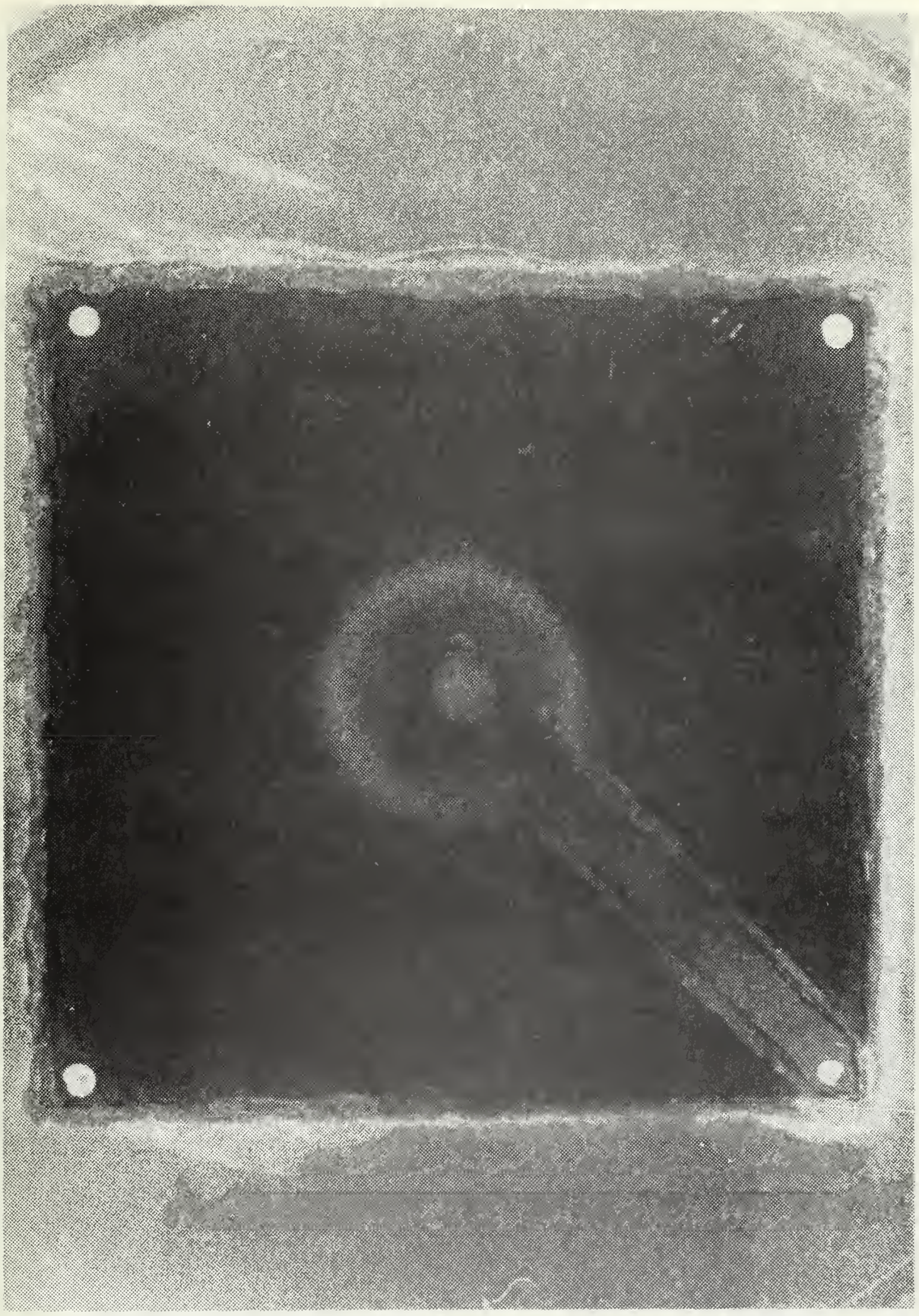

Figure 16. Radial Temperature Field for τ = 10.0.

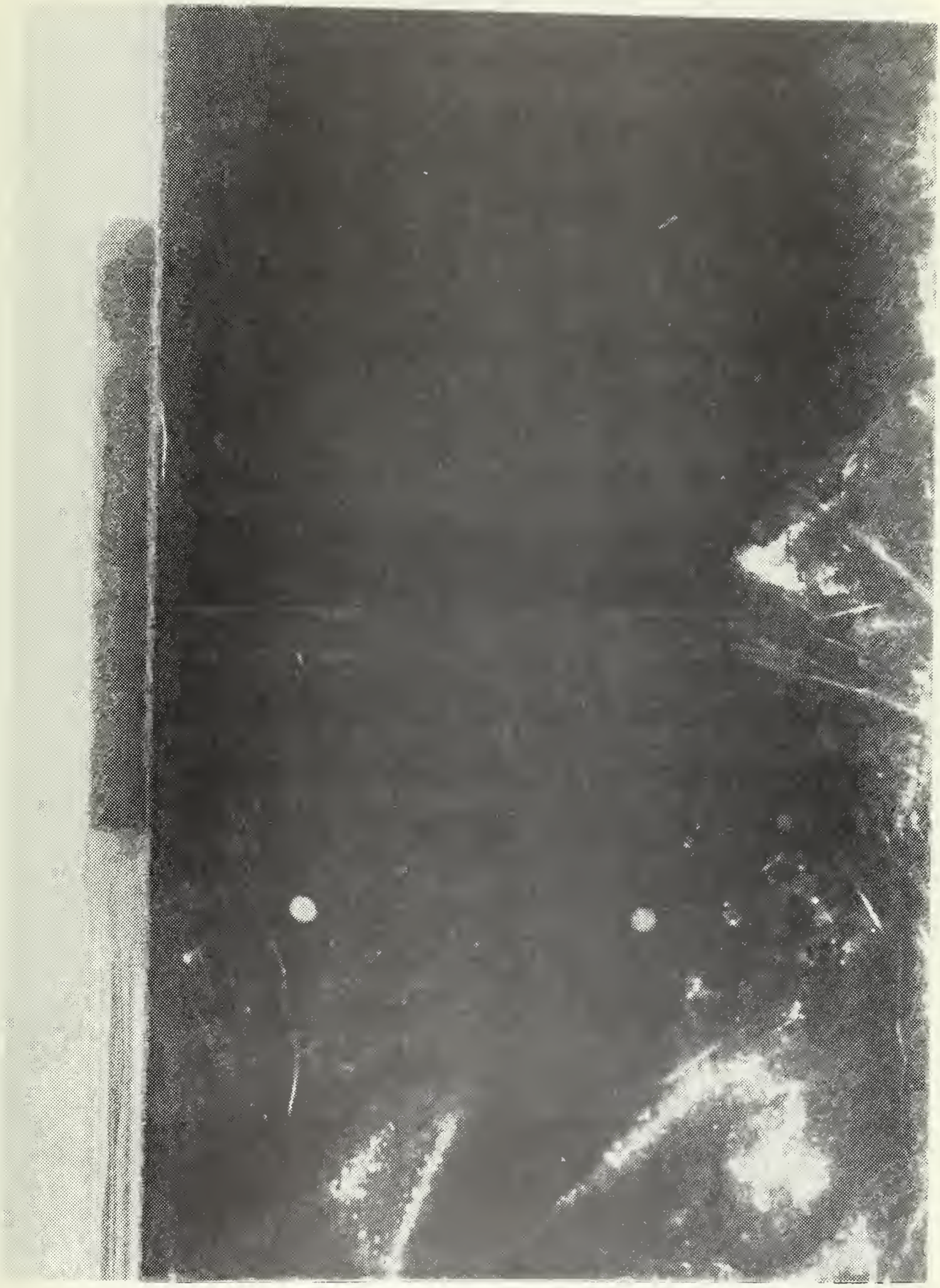

Figure 17. Axial Temperature Field for $\tau = 0.0$.

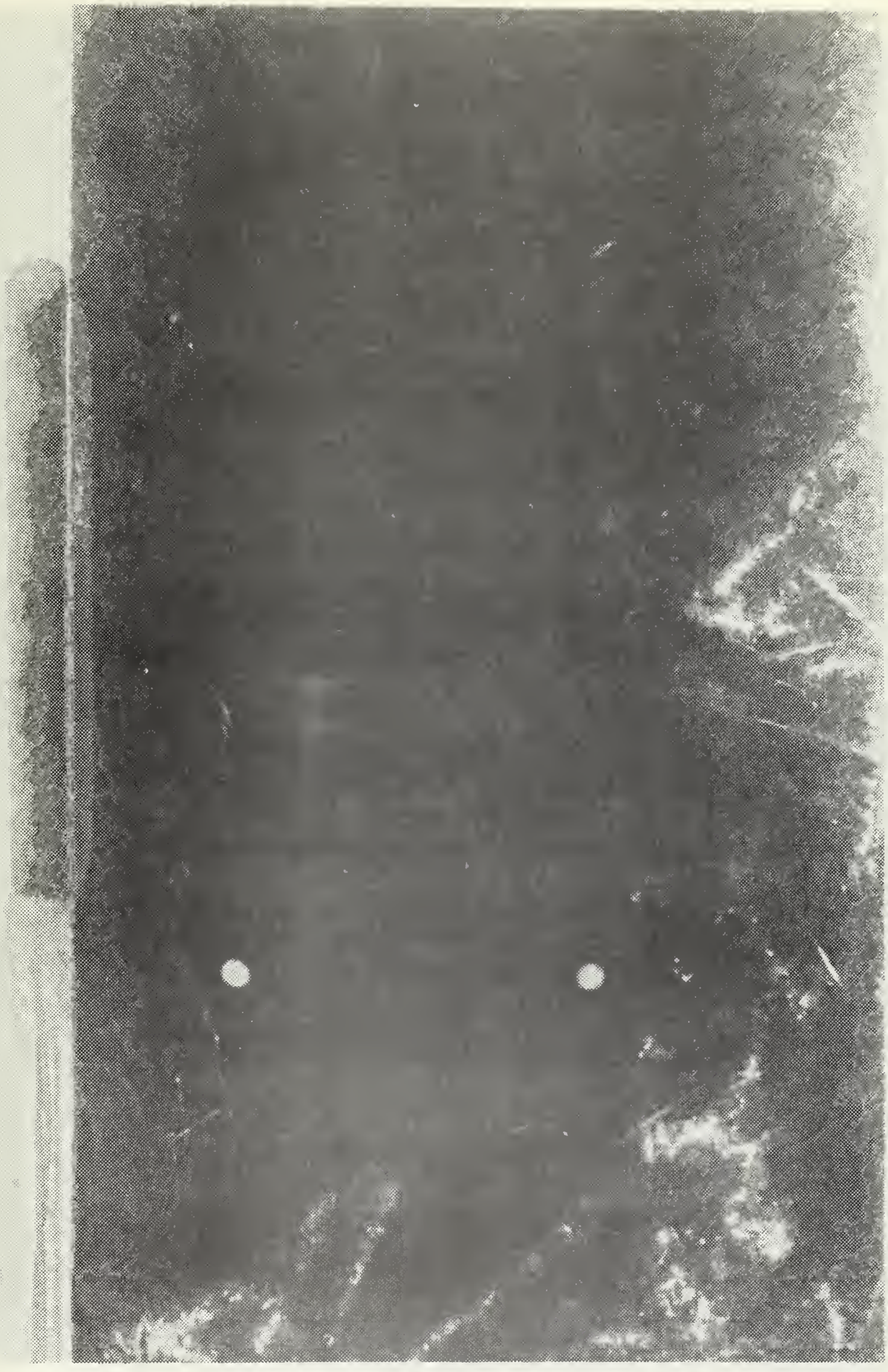

Figure 18. Axial Temperature Field for τ = 1.0.

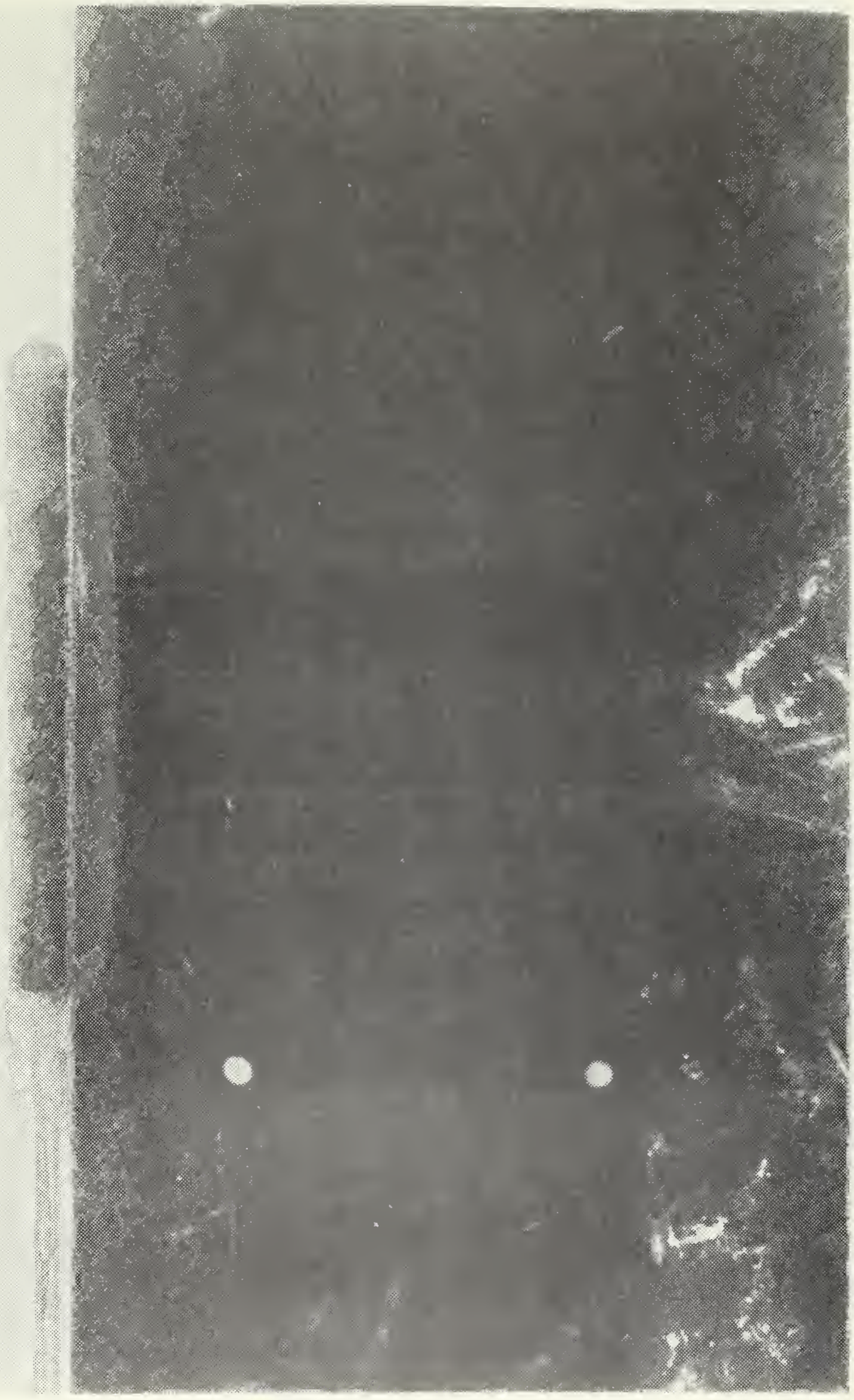

Figure 19. Axial Temperature Field for $\tau = 3.0$.

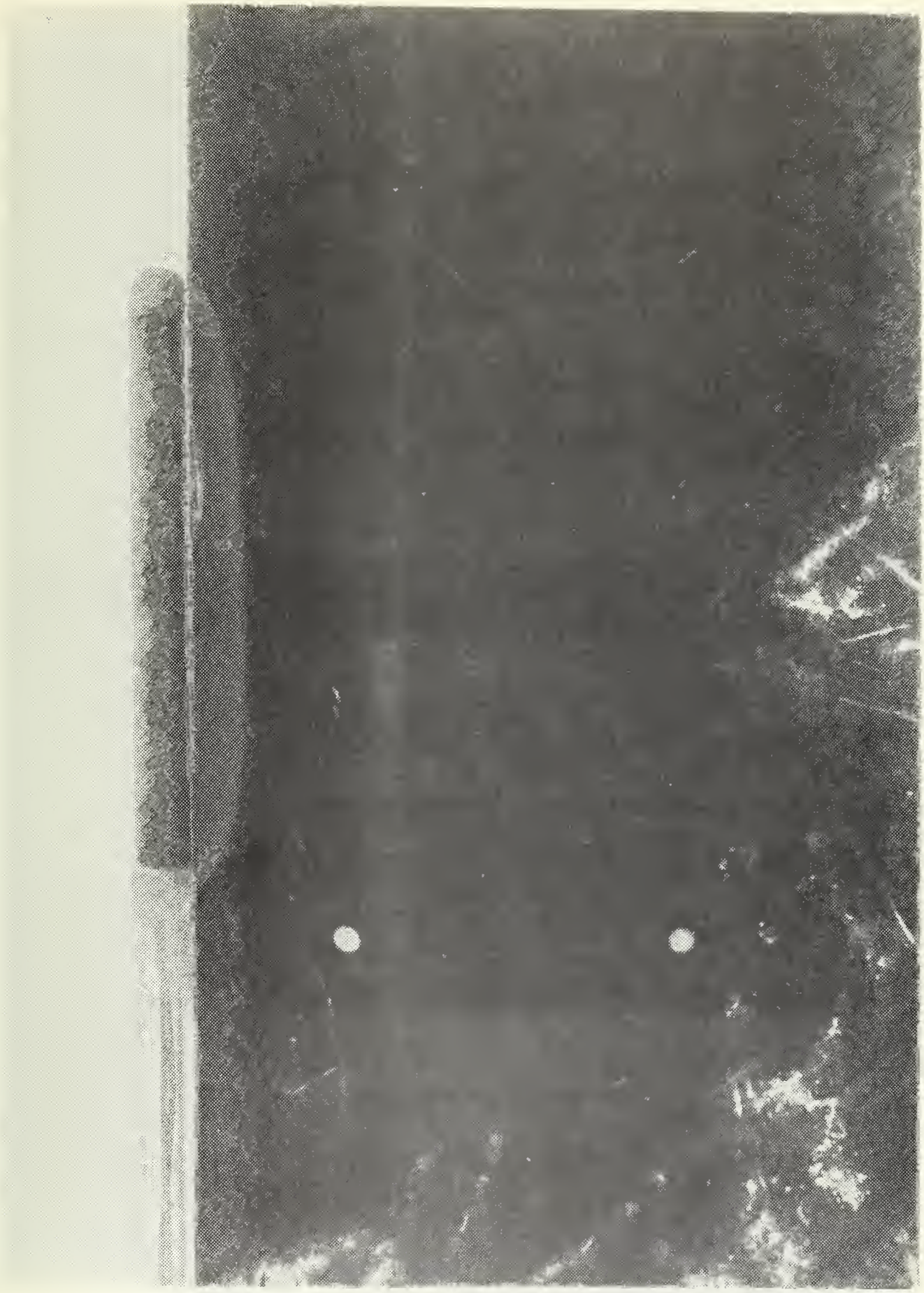

Figure 20. Axial Temperature Field for τ = 5.0.

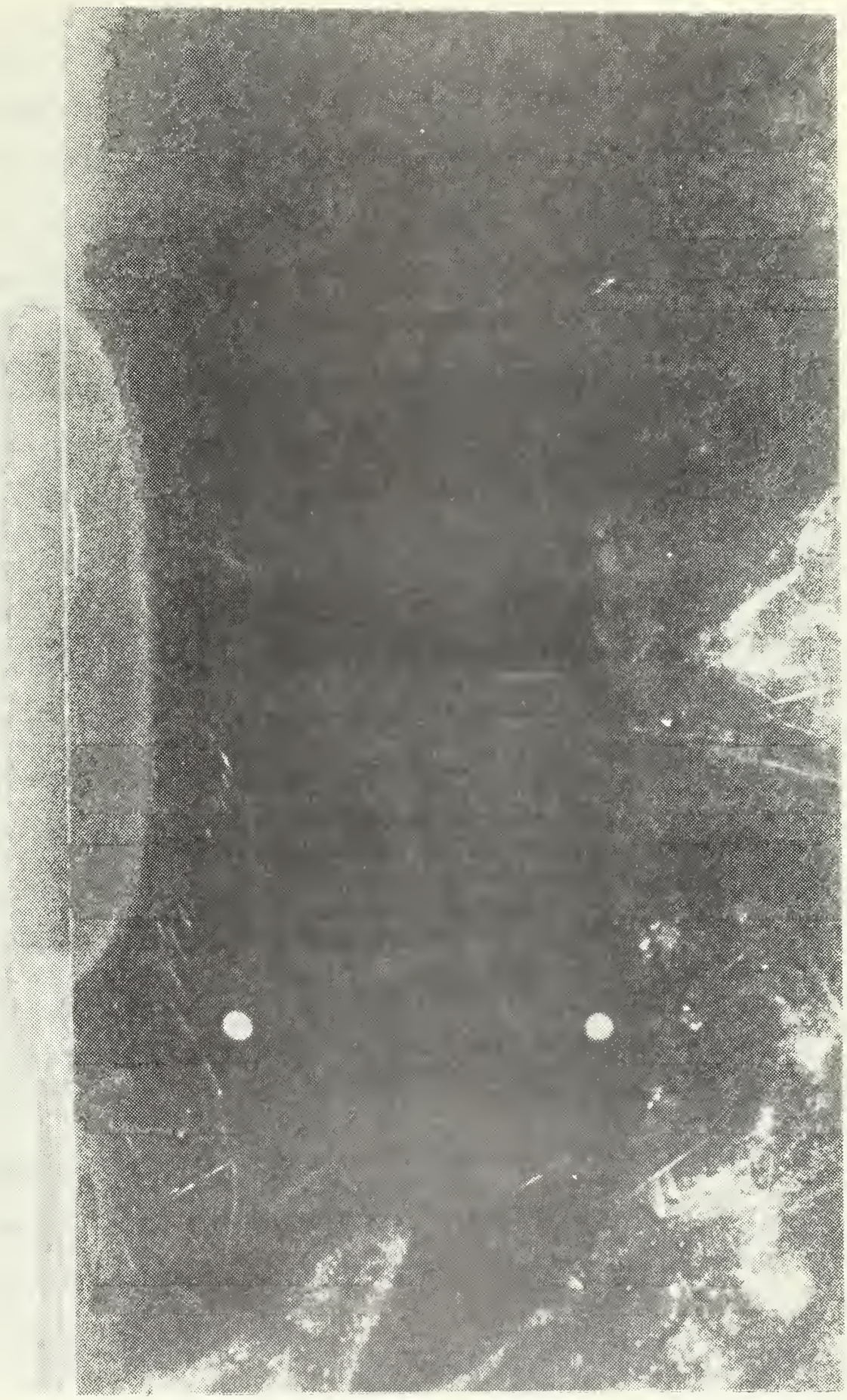

Figure 21. Axial Temperature Field for τ = 10.0.

The experimental results were compared with the two-dimensional model. Figure 22 is a typical graph of the nondimensional time for a critical temperature, $\theta_c = 0.200$. The solid line represents the two-dimensional solution and the dashed lines represent the error bound. The reader is referred to Appendix B for details of the error analysis.

The experimental results fall within the error bounds except at short times. This is due to the fact that the analytical solutions were obtained assuming a step input. That is, it was assumed that the probe came to its final temperature immediately. Because of equipment limitations this was impossible to duplicate in the experimental runs and it took the probe approximately 15 seconds to reach its final temperature.

Figure 23 is a graph of the nondimensional lesion radius versus nondimensional time for a critical temperature of $\theta_c = 0.235$. The solid line represents the two-dimensional solution.

The axial temperature field is shown in Figure 24. Again, the solid lines represent the two-dimensional solution. Only one experimental run was conducted using this configuration. Although the experimental isotherms exhibit the same shape as the analytical isotherms, they are lagging in time. A great deal of difficulty was encountered in placing the liquid crystal sheet in a uniform plane next to the probe and, in fact, the edges of the sheet tended to sag after a short period in the agar-water solution. More work is needed on this particular aspect of the experiment since experimental errors are obviously large.

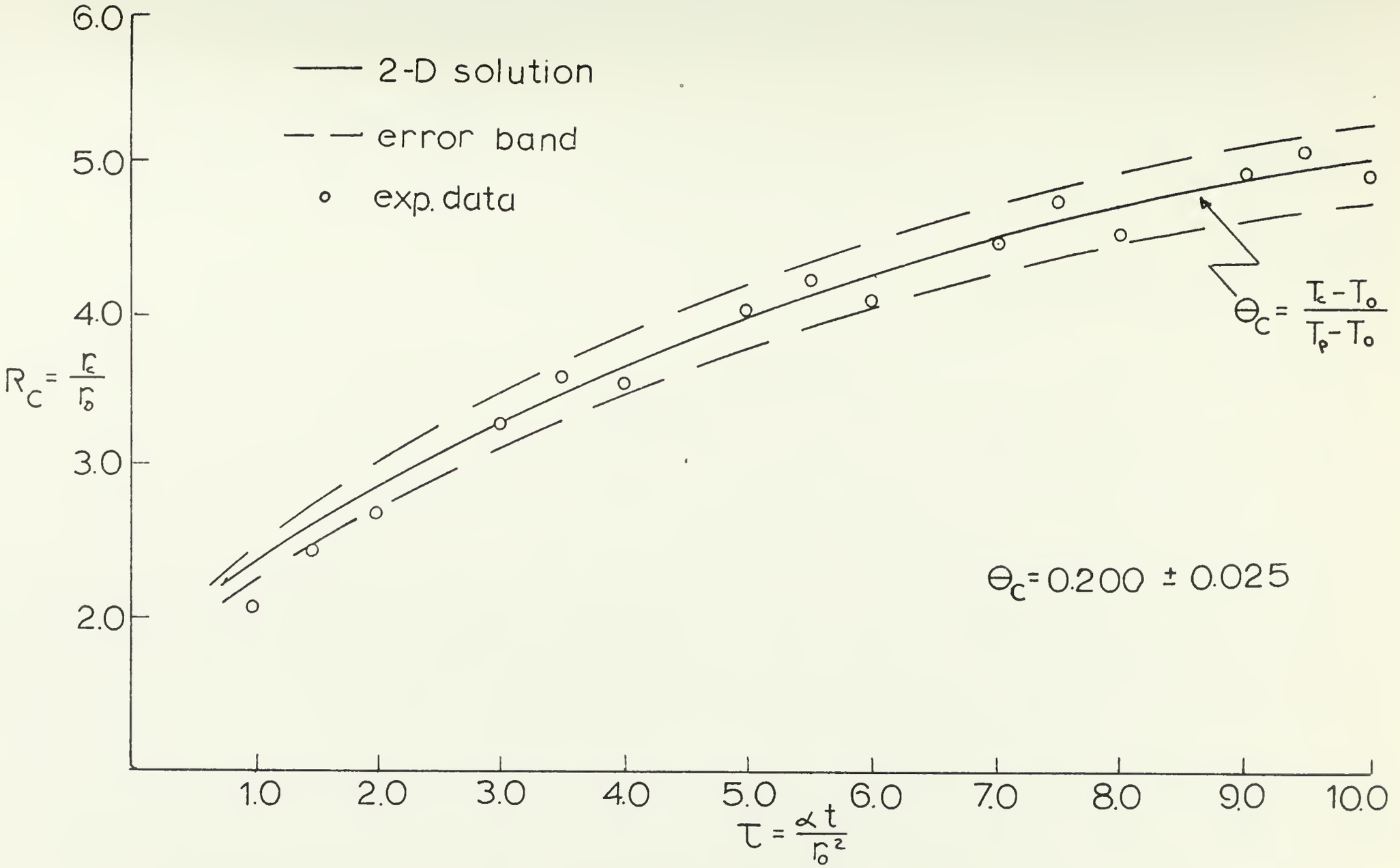

Figure 22. R_C vs τ for $\theta_C = 0.200$ and $\beta = 0.0$.

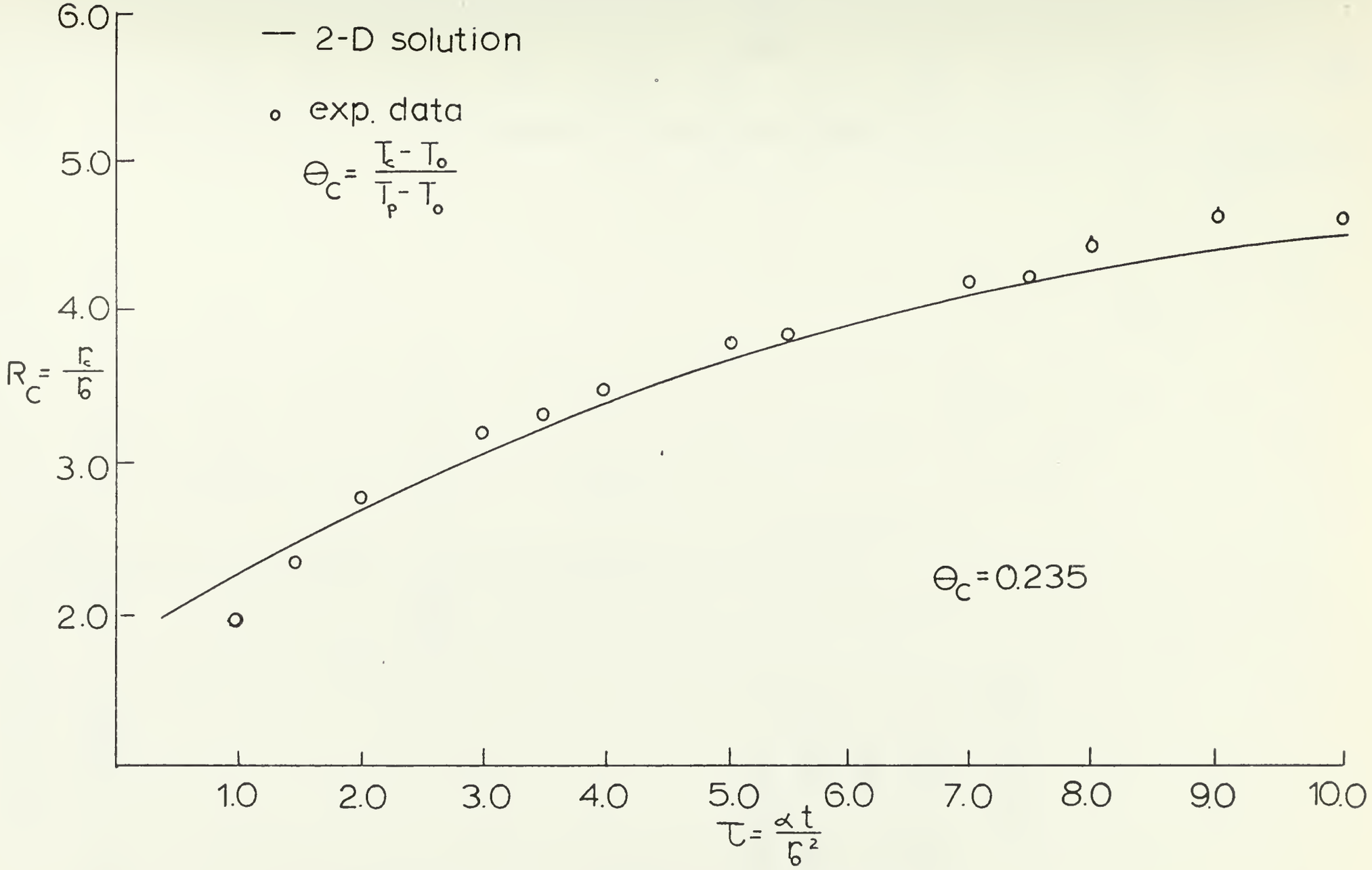

Figure 23. R_c vs τ for $\theta_c = 0.235$ and $\beta = 0.0$.

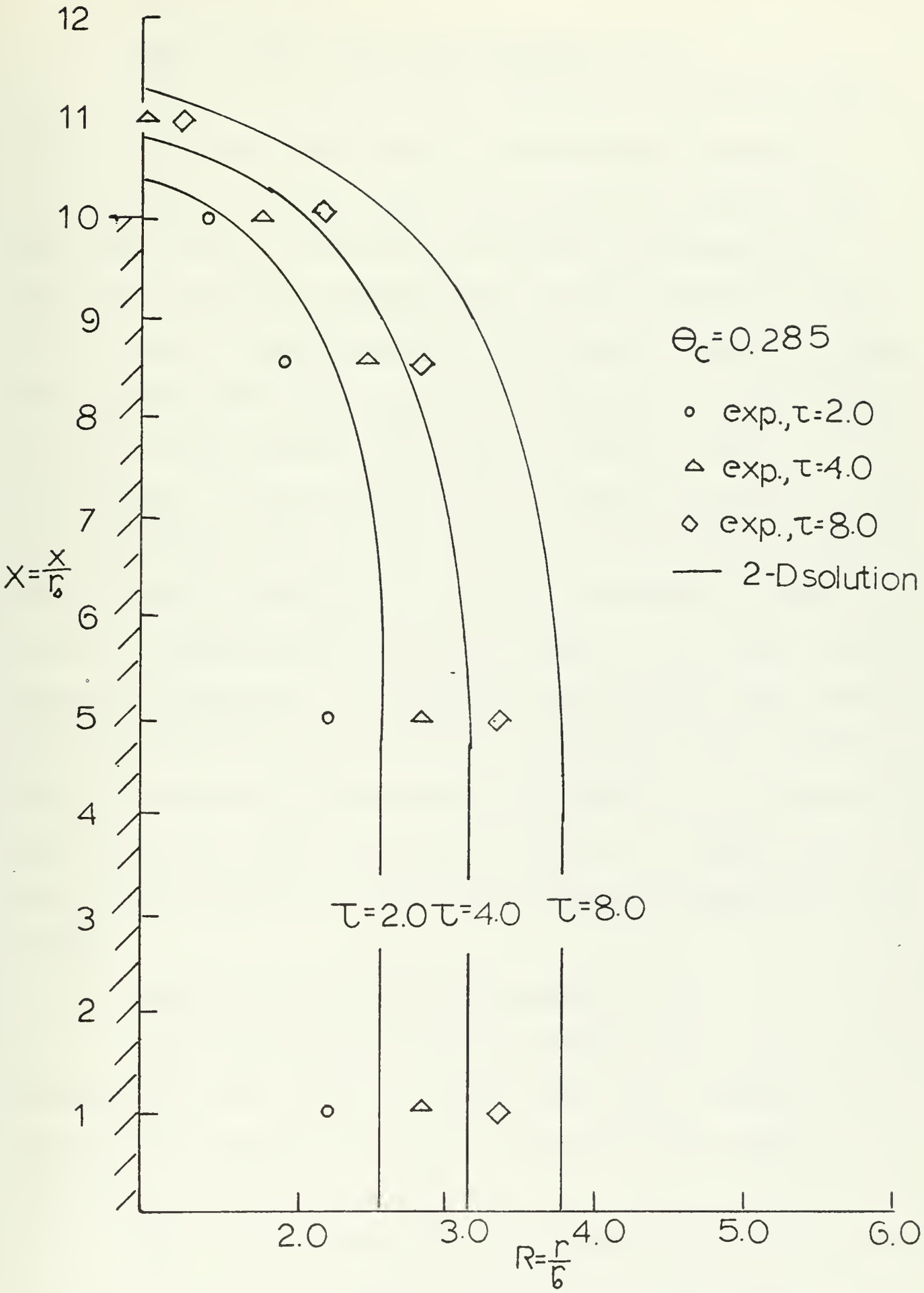

Figure 24. Axial Position (x) vs. R_c for Various Values of τ and $\theta_c = 0.285$.

B. ONE-DIMENSIONAL MODEL

1. Effect of Blood Flow on the Temperature Field

Figure 25 is a graph of the nondimensional temperature versus the radial location in the tissue for a nondimensional time equal to five (corresponds to a real time of approximately 150 seconds for the experimental probe). Four curves are presented: the dashed line represents no blood flow effect and the solid lines represent β's of 0.1, 0.5 and 1.0. Note from the graph that blood flow tends to decrease the temperature field, resulting in a slightly smaller lesion. To gain a feeling for the magnitude of this effect, a typical value of β will be calculated. A typical value for $m_b\ c_b$ of blood flow in the brain is 0.01 cal/cm^3/sec/°C [21] and a typical value of the thermal conductivity of brain tissue is 0.0014 cal/cm/°C/sec [7]. Therefore, for brain tissue, β is approximately equal to $7\ r_o^2$, where r_o is the probe radius expressed in centimeters. The experimental heating probe had a radius of approximately 0.2 cm; thus, β would be of the order of 0.28. However, an actual probe would be approximately one fourth the size of the experimental probe and β would then be much less than 0.1. Figure 26 shows that values for blood flow in this range decreased the temperature only slightly.

2. Effect of Increased Time on Temperature Field

Figure 26 is a graph of the nondimensional temperature versus nondimensional time. One can see that at short times, the rate of lesion growth is large, but as time increases, the lesion growth decreases and approaches a steady size.

Figure 27 illustrates the effect of blood flow at various times. In this graph the temperature at a radius of 2.5 is plotted against time for β = 0.0, 0.1, 0.5 and 1.0. At short times the blood flow effect

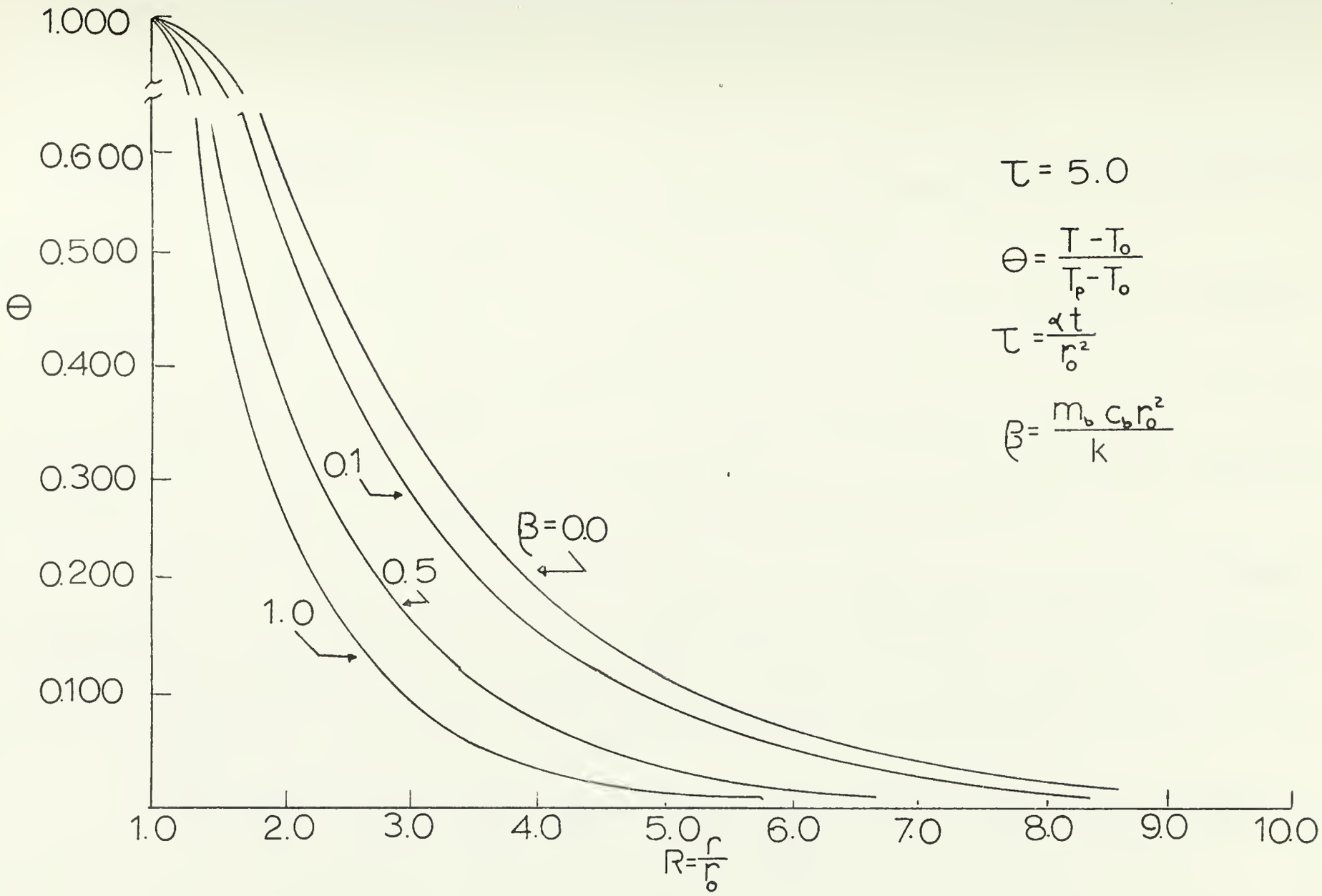

Figure 25. θ vs. R for τ = 5.0 and β = 0.0, 0.1, 0.5 and 1.0.

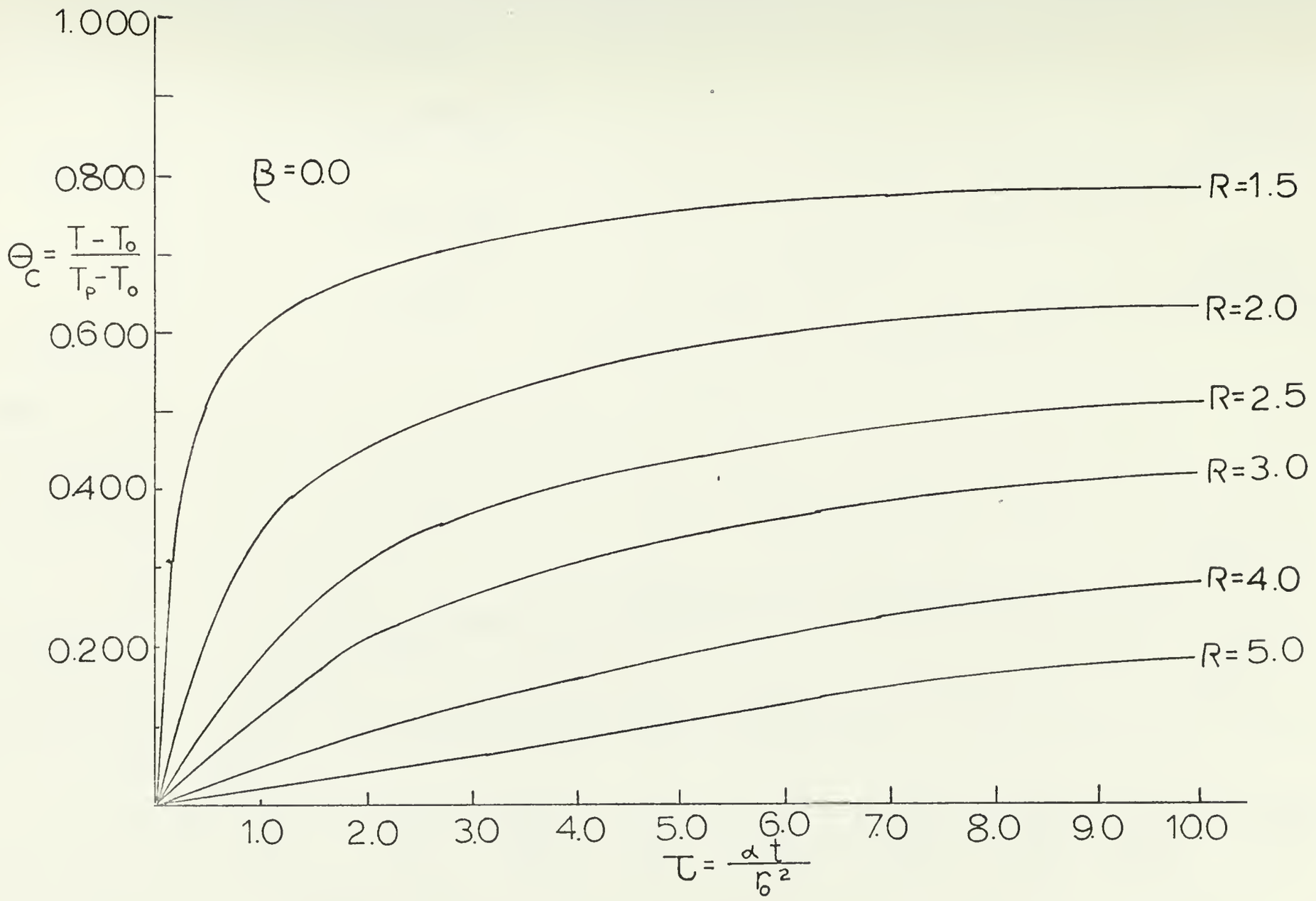

Figure 26. θ vs τ for Various Values of R and β = 0.0.

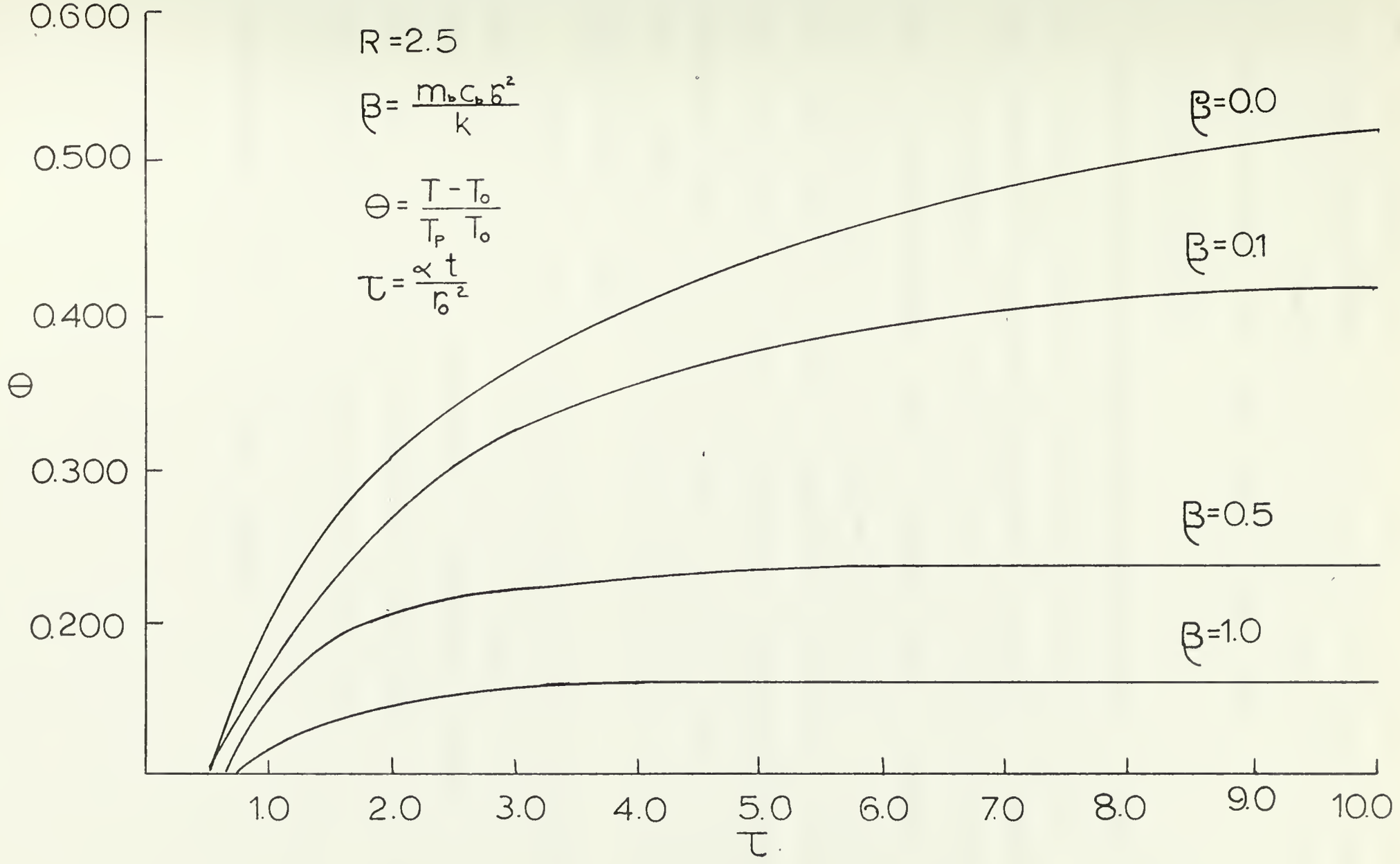

Figure 27. θ vs τ for various values of β and R = 2.5.

seems to be negligible but as time is increased, the effect of blood flow becomes more noticeable. Since the probe radius enters the blood flow parameter as a squared term, the blood flow effect would become an important parameter for a larger probe at longer times.

C. TWO-DIMENSIONAL MODEL

Results for the two-dimensional model are tabulated in Tables XII - XVI. The results are given for various positions along the probe length, where the X=0 position corresponds to the midpoint of the probe.

The two-dimensional solution was used to compare the experimental results with the theoretical results. Figure 24 shows the isotherms along the probe length for various nondimensional times. By extending this graph it can be seen that the temperature field around a cylindrical probe has an elliptical shape. The dimensions of the ellipsoid can be changed by varying the length to diameter ratio of the probe. If this ratio is made small enough , the shape of the temperature field will approach that of a sphere.

Figure 28 is a graph of the axial position (X) versus the radial location in the tissue for a nondimensional time equal to five and θ_c = 0.300. The solid lines represent the two-dimensional solution for an L/D of ten, and the dashed line represents the one-dimensional solution. Around the midpoint of the probe, the two solutions are in close agreement. However, as the distance from the mid-point increases, the one-dimensional solution is no longer accurate.

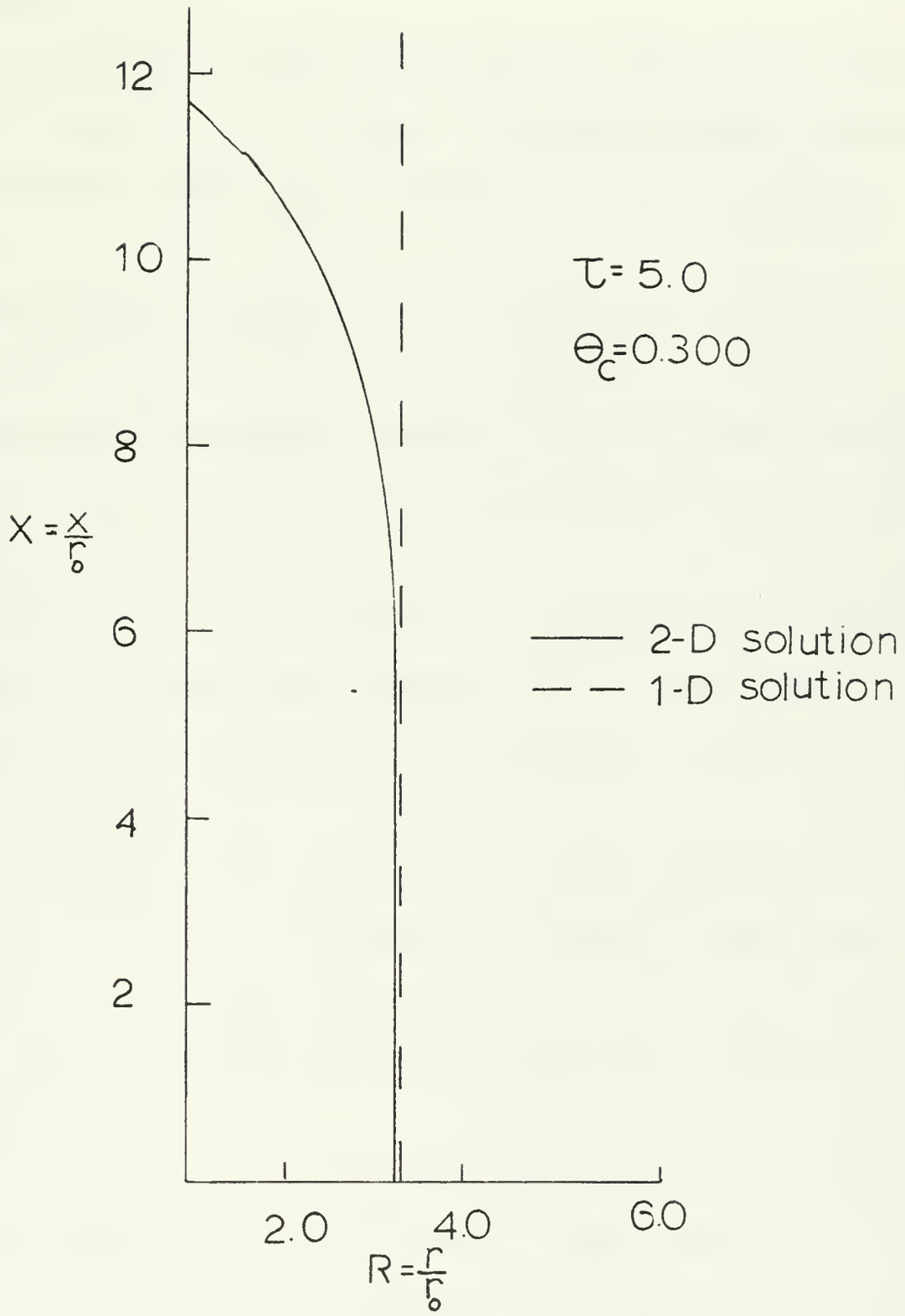

Figure 28. Axial Position (X) vs R for τ = 5.0 and θ = 0.300.

VIII. CONCLUSIONS AND RECOMMENDATIONS

A. CONCLUSIONS

Previous experimental methods used to predict lesion size and shape have generally reduced to trial and error. The usual procedure has been to produce lesions in animals, vary the time and current, and analyze the results.

The theoretical predictions of the temperature field obtained from the two-dimensional heat transfer model of the probe fall well within the error bounds of the experimental results. Thus, it appears possible to analytically predict the size of lesions produced with the constant temperature resistance probe.

Experimental isotherms are reproducible and exhibit sharp delimitation, particularly at short times. The probe itself is safe, easy to construct and simple. The temperature control circuit is small, accurate and easy to operate.

All results are presented in nondimensional form. However, it should be noted that both the experimental and theoretical models are simplified and the results have been verified only over a narrow temperature range and only for a cylindrical probe with a length to diameter ratio of ten.

The blood flow effect is found to be an important parameter in determining the temperature field in tissue if a large probe is used. For very small probes, the effect is marginal. Including the effect of blood tends to slightly decrease the temperature throughout the field and a steady state or equilibrium is reached at an earlier time than if blood flow is neglected. As time is increased, the effect of blood

flow becomes more predominant. Thus, if heat is applied for any substantial period of time, the effect of blood flow is felt more strongly.

B. RECOMMENDATIONS

It is recommended that additional experimental work be carried out with the heating probe over higher temperature ranges and with various probe dimensions and L/D ratios.

As mentioned previously, problems were encountered in attempting to place the liquid crystal tape uniformly along the axis of the probe. Accurate axial isotherms were therefore not obtained in the present set of experiments. Further investigation is needed on this aspect of the experiment.

It is also recommended that a technique for automatically controlling the probe surface temperature be developed. Two highly desirable features of such a control system would be: 1) an ability to select a range of probe surface temperatures, and 2) an ability to pre-select the duration of the heating period. These two features, together with the analytical model, will allow a more accurate prediction of lesion size.

APPENDIX A: ONE-DIMENSIONAL MODEL SOLUTION

Statement of Problem:

A cylindrical probe of radius r_o is embedded in tissue which is initially at a uniform temperature T_o. Within the tissue there is a volumetric heat generation due to metabolism of strength S_m. In addition, there is blood flow in the medium with a flowrate m_b, specific heat c_b and temperature T_b. Assume that the length to diameter ratio of the probe is large enough so that end effects may be neglected and assume the tissue is homogeneous and infinite in extent. The probe is at a constant temperature T_p. Also, assume that at an infinite distance from the probe, no temperature change occurs. Solve for the tissue temperature assuming constant properties and one dimensional radial heat flow.

Notation:

Symbol	Description	Typical Units
α	Tissue thermal diffusivity	cm^2/sec
c	Tissue specific heat	cal/gm/°C
c_b	Blood specific heat	cal/gm/°C
k	Thermal conductivity of tissue	cal/cm/sec/°C
m_b	Volumetric blood flowrate	$gm/cm^3/sec$
r	Radial location in tissue	cm
r_o	Probe radius	cm
S_m	Metabolic heat generation rate	$cal/cm^3/sec$
t	Time	sec
T	Tissue temperature	°C

Symbol	Description	Typical Units
T_b	Arterial temperature	°C
T_o	Initial tissue temperature	°C
T_p	Probe temperature	°C

Nondimensional Groups:

$$\theta = \frac{T - T_o}{T_p - T_o} \quad \text{(temperature)}$$

$$R = \frac{r}{r_o} \quad \text{(distance)}$$

$$\tau = \frac{\alpha t}{r_o^2} \quad \text{(time)}$$

$$\beta = \frac{m_b\, c_b\, r_o^2}{k} \quad \text{(blood flow)}$$

$$T_o = T_b + \frac{S_m}{m_b\, c_b} \quad \text{(initial temperature)}$$

The general heat conduction equation in cylindrical coordinates is:

$$\frac{k}{r}\,\frac{\partial\left(r\,\frac{\partial T}{\partial r}\right)}{\partial R} + m_b\, c_b\, (T_b - T) + S_m = \rho c\,\frac{\partial T}{\partial t} \qquad \text{(A.1)}$$

Substituting the above nondimensional quantities into equation (A.1) yields:

$$\frac{1}{R}\,\frac{\partial\left(R\,\frac{\partial \theta}{\partial R}\right)}{\partial R} - \beta\,\theta = \frac{\partial \theta}{\partial \tau} \qquad \text{(A.2)}$$

Equation (A.2) must be solved in order to describe the temperature of the tissue as a function of time. In order to solve the problem, boundary conditions and initial conditions must be stated. They are as follows:

(a) at $r = r_o$, (at the probe surface), $T = T_p$ or, in nondimensional form at $R = 1$, $\theta = 1$,

(b) as $r \to \infty$, the medium remains unaffected as $r \to \infty$, $T \to T_o$ or, in nondimensional form as $R \to \infty$, $\theta \to o$,

(c) at $t = o$, $T = T_o$ or, in nondimensional form at $\tau = o$, $\theta = o$.

By using Laplace Transformations, a closed form solution to equation (A.2) can be obtained:

$$\theta = \frac{K_o \sqrt{\beta R}}{K_o \sqrt{\beta}} - \frac{2}{\pi} \int_o^\infty \frac{e^{-\tau u^2} C_o (u, Ru)}{\left(u + \frac{\beta}{u}\right)[Y_o^2 (u) + Y_o^2 (u)]} du \qquad (A.3)$$

$$C_o = J_o (u) Y_o (uR) - J_o (uR) Y_o (u)$$

If the blood flow term is neglected, equation (A.3) reduces to:

$$\theta = 1 - \frac{2}{\pi} \int_o^\infty \frac{e^{-\tau u^2} C_o (u, Ru)\, du}{u [J_o^2 (u) + Y_o^2 (u)]} \qquad (A.4)$$

Equation (A.4) is the closed form solution given by Carslaw and Jaeger [5]. This integral has been solved by Jaeger [15].

Solution:

A finite increment technique was used to solve for the temperature field. Equation (A.3) was rewritten in the following form:

$$\theta = \frac{K_o \left(\sqrt{\beta R}\right)}{K_o \left(\sqrt{\beta}\right)} - \frac{2}{\pi} \int_o^\infty q (u)\, du \qquad (A.5)$$

$$q (u) = \frac{e^{-\tau u^2} C_o (u, Ru)}{\left(u + \frac{\beta}{u}\right)[J_o^2 (u) + Y_o^2 (u)]}$$

Assume an approximation to the integral can be derived by passing an interpolating function through several adjacent points and then

integrating the interpolating function. Simpson's Rule uses the parabola as an interpolatory function between u_{i-1}, u_i and u_{i+1} where the u_i are evenly spaced values of the interval (u_o, u_n) with a subinterval length $h = u_{i+1} - u_i$. Integrating the parabolic interpolating function gives:

$$\int_{u_{i-1}}^{u_{i+1}} q(u)\, du \simeq \frac{h}{3} [Y_{i-1} + 4_{yi} + Y_{i+1}] \qquad (A.6)$$

where $Y_i = q(u_i)$

Expanding equation (A.6) over the entire interval yields,

$$\int_{o}^{n} q(u)\, du \simeq \frac{h}{3} [Y_o + 4Y_1 + 2Y_2 + \ldots 4Y_{n-1} + Y_n] \qquad (A.7)$$

Computer programs 1, 2 and 3 were used to solve this problem for $\beta = 0.0$. Programs 4, 5, and 6 were used to solve for $\beta = 0.1$. To solve for other values of β only one card must be changed.

Because $q(u)$ is singular at $u = 0$, the lower limit of the integral was taken as 0.001. The error caused by this was assumed to be negligible. By varying the upper limit of the integral between 1.0 and 100.0, it was found that the solution coverged at approximately 8.0. Therefore, an upper limit of 10.0 was used.

The Y Bessel functions involving the lower limit of the integral, 0.001, were solved using a separate program and calling the BESY subroutine. This was done to avoid underflow and reduce computer time.

The included programs solve the temperature field for times between 0.1 and 1.0 only. To change the time interval only one card must be changed.

Jaeger's results [15] were used as a check on the accuracy of the results obtained from Programs 1, 2 and 3 when $\beta = 0.0$. The greatest difference between the solutions was 0.001.

APPENDIX B: ERROR ANALYSIS OF EXPERIMENTAL RESULTS

Three nondimensional quantities were calculated in analyzing the experimental results.

$$\theta_c = \frac{T_c - T_o}{T_p - T_o} \quad \text{(critical temperature)}$$

$$R_c = \frac{r_c}{r_o} \quad \text{(critical radius)}$$

$$\tau = \frac{\alpha\, t}{r_o^2} \quad \text{(time)}$$

Ambrosius [1] gives the following equations for the resultant errors (R) in computations involving measurements:

(a) addition

$$(A \pm a) + (B \pm b)$$

$$R = \pm (a^2 + b^2)^{\frac{1}{2}} \tag{B.1}$$

(b) subtraction

$$(A \pm a) = (B \pm b)$$

$$R = \pm (a^2 + b^2)^{\frac{1}{2}} \tag{B.2}$$

(c) multiplication

$$(A \pm a)\ (B \pm b)$$

$$R = \pm [(Ab)^2 + (Ba)^2]^{\frac{1}{2}} \tag{B.3}$$

(d) division

$$(B \pm b) \,/\, (A \pm a)$$

$$R = \pm \left[\frac{(Ba/A)^2 + (b)^2}{(A)^2}\right]^{\frac{1}{2}} \tag{B.4}$$

where A and B are two measurements and an error of ± a is made in the measurement of A and an error of ± b is made in the measurement of B

The following errors were estimated for the variables in θ_c:

$$T_c = \pm 0.1°C$$

$$T_o = \pm 0.3°C$$

$$T_p = \pm 0.2°C$$

An examination of θ_c along with equation (B.4) indicates that the largest error in θ_c occurs when T_o is the greatest, or when the quantity $T_c - T_o$ is the smallest. The highest value of T_o was in test run 10. The resultant errors will now be estimated using data from run 10.

$$T_o = 25.2°C \pm 0.3°C$$

$$T_c = 29.9°C \pm 0.1°C$$

$$T_p = 48.5°C \pm 0.2°C$$

$$T_c - T_o = 4.7°C$$

$$T_p - T_o = 18.6°C$$

Substituting these values into equation (B.2), the following errors are obtained:

$$T_c - T_o = \pm \ 0.32°C$$

$$T_p - T_o + \pm \ 0.36°C$$

From equation (B.4) the resultant error in θ_c is:

$$R = \pm 0.025$$

or, the percentage error in θ_c = 12.5%. This error would be reduced if a higher probe temperature (T_p) or a higher critical temperature (T_c) were used.

An analysis of the error in computing R_c and τ is more difficult since the variables in these quantities change with time. It was estimated that at short times the error in both R_c and τ could be of the order of 10%. However, this error decreases significantly as time is increased.

TABLE II

VALUES OF Θ FOR NONDIMENSIONAL TIME(τ) AND NONDIMENSIONAL LOCATION(R), ONE-DIMENSIONAL MODEL.

$\tau = 0.1 \rightarrow 0.9$
$R = 1.0 \rightarrow 10.0$
$\beta = 0.0$

τ / R	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.2	0.601	0.691	0.733	0.758	0.776	0.789	0.800	0.808	0.815
1.4	0.316	0.450	0.519	0.562	0.592	0.615	0.634	0.648	0.661
1.6	0.143	0.274	0.352	0.405	0.443	0.472	0.496	0.515	0.532
1.8	0.055	0.156	0.229	0.282	0.323	0.356	0.382	0.405	0.424
2.0	0.018	0.082	0.142	0.190	0.230	0.263	0.290	0.313	0.333
2.2	0.005	0.039	0.083	0.124	0.159	0.189	0.216	0.239	0.259
2.4	0.001	0.017	0.046	0.077	0.107	0.134	0.158	0.179	0.199
2.6	0.000	0.007	0.024	0.047	0.070	0.092	0.113	0.132	0.150
2.8		0.002	0.012	0.027	0.044	0.062	0.079	0.096	0.111
3.0		0.001	0.006	0.015	0.027	0.040	0.054	0.068	0.081
3.2		0.000	0.002	0.008	0.016	0.025	0.036	0.047	0.058
3.4			0.001	0.004	0.009	0.016	0.024	0.032	0.041
3.6			0.000	0.002	0.005	0.009	0.015	0.021	0.029
3.8				0.001	0.002	0.005	0.009	0.014	0.019
4.0				0.000	0.001	0.003	0.006	0.009	0.013
5.0					0.000	0.000	0.000	0.001	0.001
6.0								0.000	0.000
7.0									
8.0									
9.0									
10.0									

TABLE III

VALUES OF Θ FOR NONDIMENSIONAL TIME(τ) AND NONDIMENSIONAL LOCATION(R), ONE-DIMENSIONAL MODEL.

$\tau = 1.0 \rightarrow 9.0$
$R = 1.0 \rightarrow 10.0$
$\beta = 0.0$

R \ τ	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0
1.2	0.821	0.854	0.869	0.879	0.885	0.890	0.894	0.898	0.900
1.4	0.672	0.732	0.766	0.777	0.789	0.798	0.805	0.811	0.816
1.6	0.546	0.627	0.665	0.689	0.706	0.718	0.728	0.736	0.743
1.8	0.440	0.536	0.583	0.612	0.633	0.648	0.661	0.671	0.679
2.0	0.351	0.458	0.511	0.544	0.568	0.586	0.601	0.612	0.632
2.2	0.277	0.389	0.447	0.484	0.510	0.531	0.547	0.560	0.571
2.4	0.216	0.330	0.390	0.430	0.459	0.481	0.498	0.513	0.525
2.6	0.166	0.278	0.340	0.382	0.412	0.435	0.454	0.470	0.483
2.8	0.126	0.233	0.296	0.338	0.370	0.394	0.414	0.430	0.444
3.0	0.094	0.194	0.256	0.299	0.332	0.357	0.377	0.394	0.409
3.2	0.069	0.161	0.221	0.264	0.297	0.322	0.344	0.361	0.377
3.4	0.050	0.132	0.190	0.233	0.265	0.292	0.313	0.331	0.347
3.6	0.036	0.108	0.163	0.205	0.237	0.263	0.285	0.303	0.319
3.8	0.025	0.088	0.139	0.179	0.211	0.237	0.259	0.277	0.292
4.0	0.017	0.071	0.118	0.157	0.188	0.213	0.235	0.253	0.269
5.0	0.002	0.021	0.049	0.076	0.101	0.122	0.142	0.159	0.174
6.0	0.000	0.005	0.018	0.034	0.051	0.067	0.082	0.096	0.110
7.0		0.001	0.005	0.013	0.023	0.034	0.045	0.056	0.067
8.0		0.000	0.001	0.005	0.010	0.016	0.024	0.031	0.039
9.0			0.000	0.001	0.004	0.007	0.012	0.016	0.022
10.0				0.000	0.001	0.003	0.005	0.008	0.012

TABLE IV

VALUES OF Θ FOR NONDIMENSIONAL TIME(τ) AND NONDIMENSIONAL LOCATION(R), ONE-DIMENSIONAL MODEL.

τ = 10.0 → 100.0
R = 1.0 → 10.0
β = 0.0

τ / R	10.0	20.0	30.0	40.0	50.0	70.0	100.0
1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.2	0.903	0.916	0.922	0.926	0.929	0.933	0.937
1.4	0.820	0.845	0.857	0.864	0.869	0.877	0.884
1.6	0.749	0.783	0.800	0.810	0.818	0.828	0.838
1.8	0.687	0.729	0.750	0.763	0.772	0.785	0.797
2.0	0.631	0.681	0.705	0.720	0.731	0.746	0.760
2.2	0.581	0.637	0.664	0.682	0.694	0.711	0.728
2.4	0.536	0.597	0.628	0.647	0.660	0.679	0.698
2.6	0.494	0.561	0.594	0.615	0.629	0.650	0.670
2.8	0.457	0.529	0.563	0.585	0.601	0.623	0.644
3.0	0.422	0.497	0.534	0.557	0.574	0.598	0.621
3.2	0.390	0.468	0.507	0.532	0.550	0.575	0.598
3.4	0.360	0.441	0.482	0.508	0.526	0.553	0.578
3.6	0.333	0.416	0.458	0.485	0.505	0.532	0.558
3.8	0.307	0.392	0.436	0.464	0.484	0.512	0.539
4.0	0.283	0.370	0.415	0.444	0.465	0.494	0.522
5.0	0.188	0.277	0.325	0.358	0.381	0.414	0.446
6.0	0.122	0.207	0.256	0.290	0.315	0.350	0.385
7.0	0.077	0.153	0.201	0.235	0.260	0.297	0.334
8.0	0.047	0.112	0.157	0.190	0.216	0.253	0.291
9.0	0.027	0.081	0.122	0.153	0.178	0.215	0.254
10.0	0.015	0.057	0.094	0.123	0.147	0.183	0.222
20.0	0.000	0.001	0.003	0.009	0.016	0.031	0.053
30.0		0.000	0.000	0.000	0.001	0.003	0.010
40.0					0.000	0.000	0.002
50.0							0.000

TABLE V

VALUES OF Θ FOR NONDIMENSIONAL TIME(τ) AND NONDIMENSIONAL LOCATION(R), ONE-DIMENSIONAL MODEL.

$\tau = 0.1 \rightarrow 0.9$
$R = 1.0 \rightarrow 10.0$
$\beta = 0.1$

τ / R	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.2	0.599	0.688	0.729	0.754	0.770	0.783	0.793	0.800	0.807
1.4	0.314	0.447	0.514	0.555	0.584	0.606	0.623	0.637	0.649
1.6	0.142	0.272	0.348	0.398	0.435	0.463	0.485	0.503	0.518
1.8	0.055	0.154	0.225	0.277	0.316	0.347	0.372	0.392	0.410
2.0	0.018	0.081	0.139	0.186	0.224	0.255	0.281	0.302	0.321
2.2	0.005	0.039	0.082	0.121	0.155	0.183	0.208	0.229	0.248
2.4	0.001	0.017	0.046	0.076	0.104	0.129	0.151	0.171	0.189
2.6	0.000	0.007	0.024	0.045	0.067	0.089	0.108	0.126	0.142
2.8		0.003	0.012	0.026	0.043	0.059	0.076	0.091	0.105
3.0		0.001	0.006	0.014	0.026	0.039	0.052	0.064	0.077
3.2		0.000	0.002	0.008	0.015	0.025	0.035	0.045	0.055
3.4			0.001	0.004	0.009	0.015	0.023	0.031	0.039
3.6			0.000	0.002	0.005	0.009	0.014	0.020	0.027
3.8				0.001	0.003	0.005	0.009	0.013	0.018
4.0				0.000	0.001	0.003	0.005	0.009	0.012
5.0					0.000	0.000	0.000	0.001	0.001
6.0								0.000	0.000
7.0									
8.0									
9.0									
10.0									

TABLE VI

VALUES OF Θ FOR NONDIMENSIONAL TIME(τ) AND NONDIMENSIONAL LOCATION(R), ONE-DIMENSIONAL MODEL.

$\tau = 1.0 \rightarrow 9.0$
$R = 1.0 \rightarrow 10.0$
$\beta = 0.1$

R \ τ	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0
1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.2	0.812	0.841	0.853	0.860	0.864	0.867	0.869	0.870	0.872
1.4	0.658	0.710	0.732	0.744	0.752	0.757	0.761	0.764	0.766
1.6	0.531	0.601	0.631	0.648	0.658	0.666	0.671	0.675	0.678
1.8	0.425	0.508	0.545	0.565	0.579	0.588	0.594	0.599	0.603
2.0	0.337	0.429	0.471	0.494	0.510	0.520	0.528	0.533	0.538
2.2	0.265	0.362	0.407	0.433	0.450	0.462	0.470	0.476	0.481
2.4	0.205	0.304	0.351	0.379	0.397	0.410	0.419	0.426	0.432
2.6	0.157	0.254	0.303	0.332	0.351	0.365	0.375	0.382	0.388
2.8	0.119	0.211	0.260	0.291	0.311	0.325	0.335	0.343	0.349
3.0	0.089	0.175	0.224	0.254	0.275	0.289	0.300	0.308	0.314
3.2	0.065	0.144	0.191	0.222	0.243	0.257	0.269	0.277	0.283
3.4	0.047	0.118	0.163	0.193	0.214	0.229	0.240	0.256	
3.6	0.034	0.096	0.139	0.168	0.189	0.204	0.215	0.224	0.231
3.8	0.024	0.077	0.118	0.146	0.168	0.181	0.193	0.201	0.208
4.0	0.016	0.062	0.099	0.126	0.146	0.161	0.172	0.181	0.188
5.0	0.002	0.019	0.040	0.059	0.075	0.088	0.098	0.106	0.112
6.0	0.000	0.005	0.014	0.026	0.036	0.046	0.054	0.061	0.066
7.0		0.001	0.005	0.010	0.017	0.023	0.029	0.034	0.038
8.0		0.000	0.001	0.004	0.007	0.011	0.015	0.018	0.021
9.0			0.000	0.001	0.003	0.005	0.007	0.009	0.012
10.0				0.000	0.001	0.002	0.003	0.005	0.006

TABLE VII

VALUES OF Θ FOR NONDIMENSIONAL TIME(τ) AND NONDIMENSIONAL LOCATION(R), ONE-DIMENSIONAL MODEL.

τ=10.0→100.0
R=1.0→50.0
β=0.1

τ / R	10.0	20.0	30.0	40.0	50.0	70.0	100.0
1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.2	0.872	0.876	0.876	0.876	0.876	0.877	0.877
1.4	0.768	0.774	0.775	0.775	0.775	0.775	0.775
1.6	0.680	0.689	0.690	0.690	0.691	0.691	0.691
1.8	0.605	0.616	0.618	0.618	0.618	0.618	0.618
2.0	0.541	0.553	0.556	0.556	0.556	0.556	0.556
2.2	0.485	0.499	0.501	0.502	0.502	0.502	0.502
2.4	0.436	0.451	0.454	0.454	0.455	0.455	0.455
2.6	0.392	0.409	0.412	0.413	0.413	0.413	0.413
2.8	0.354	0.371	0.375	0.375	0.375	0.376	0.376
3.0	0.319	0.338	0.341	0.342	0.042	0.342	0.342
3.2	0.288	0.308	0.311	0.312	0.312	0.313	0.313
3.4	0.261	0.281	0.285	0.286	0.286	0.286	0.286
3.6	0.236	0.257	0.261	0.261	0.262	0.262	0.262
3.8	0.214	0.235	0.239	0.240	0.240	0.240	0.240
4.0	0.193	0.215	0.219	0.220	0.220	0.220	0.220
5.0	0.118	0.140	0.144	0.145	0.145	0.145	0.145
6.0	0.071	0.092	0.096	0.097	0.097	0.098	0.098
7.0	0.042	0.061	0.065	0.066	0.066	0.066	0.066
8.0	0.024	0.040	0.044	0.045	0.045	0.045	0.045
9.0	0.014	0.026	0.030	0.031	0.031	0.031	0.031
10.0	0.007	0.017	0.020	0.021	0.022	0.022	0.022
20.0	0.000	0.000	0.000	0.001	0.001	0.001	0.001
30.0				0.000	0.000	0.000	0.000
40.0							
50.0							

TABLE VIII

VALUES OF Θ FOR NONDIMENSIONAL TIME(τ) AND NONDIMENSIONAL LOCATION(R), ONE-DIMENSIONAL MODEL.

$\tau = 0.1 \rightarrow 0.9$
$R = 1.0 \rightarrow 10.0$
$\Theta = 0.5$

τ / R	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.2	0.592	0.676	0.713	0.735	0.749	0.759	0.766	0.772	0.777
1.4	0.308	0.433	0.494	0.530	0.555	0.572	0.585	0.596	0.604
1.6	0.139	0.261	0.330	0.374	0.404	0.427	0.444	0.457	0.468
1.8	0.053	0.147	0.211	0.256	0.289	0.314	0.333	0.348	0.361
2.0	0.017	0.076	0.129	0.170	0.202	0.227	0.247	0.263	0.276
2.2	0.005	0.037	0.075	0.110	0.138	0.161	0.180	0.196	0.209
2.4	0.001	0.016	0.042	0.068	0.091	0.112	0.129	0.144	0.157
2.6	0.000	0.007	0.022	0.041	0.059	0.076	0.091	0.104	0.116
2.8		0.002	0.011	0.023	0.037	0.050	0.063	0.074	0.085
3.0		0.001	0.005	0.013	0.022	0.033	0.043	0.052	0.061
3.2		0.000	0.002	0.007	0.013	0.021	0.028	0.036	3.043
3.4			0.001	0.003	0.007	0.013	0.018	0.024	0.030
3.6			0.000	0.002	0.004	0.008	0.012	0.016	3.021
3.8				0.001	0.002	0.004	0.007	0.010	0.014
4.0				0.000	0.001	0.002	0.004	0.007	0.009
5.0					0.000	0.000	0.000	0.001	0.001
6.0								0.000	0.000
7.0									
8.0									
9.0									
10.0									

TABLE IX

VALUES OF Θ FOR NONDIMENSIONAL TIME(τ) AND NONDIMENSIONAL LOCATION(R), ONE-DIMENSIONAL MODEL.

$\tau = 1.0 \rightarrow 9.0$
$R = 1.0 \rightarrow 10.0$
$\beta = 0.5$

τ	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0
R									
1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.2	0.780	0.797	0.801	0.803	0.804	0.804	0.804	0.834	0.804
1.4	0.610	0.640	0.648	0.652	0.653	0.653	0.654	0.654	0.654
1.6	0.477	0.517	0.528	0.532	0.534	0.535	0.536	0.536	0.536
1.8	0.371	0.419	0.432	0.438	0.440	0.441	0.441	0.442	0.442
2.0	0.287	0.340	0.355	0.361	0.364	0.365	0.366	0.366	0.366
2.2	0.220	0.276	0.293	0.299	0.302	0.302	0.304	0.304	0.304
2.4	0.167	0.223	0.241	0.248	0.251	0.253	0.254	0.254	0.254
2.6	0.126	0.181	0.199	0.207	0.210	0.211	0.212	0.213	0.213
2.8	0.094	0.146	0.165	0.172	0.176	0.177	0.178	0.178	0.179
3.0	0.069	0.118	0.136	0.144	0.147	0.149	0.149	0.150	0.150
3.2	0.050	0.094	0.112	0.120	0.123	0.125	0.126	0.126	0.126
3.4	0.036	0.075	0.092	0.100	0.104	0.105	0.106	0.106	0.107
3.6	0.025	0.060	0.078	0.083	0.087	0.089	0.090	0.090	0.090
3.8	0.018	0.047	0.063	0.070	0.073	0.075	0.076	0.076	0.076
4.0	0.017	0.037	0.051	0.058	0.061	0.063	0.064	0.064	0.065
5.0	0.001	0.010	0.018	0.023	0.026	0.027	0.028	0.028	0.029
6.0	0.000	0.002	0.006	0.009	0.011	0.012	0.012	0.013	0.013
7.0		0.000	0.002	0.003	0.004	0.005	0.005	0.006	0.006
8.0			0.000	0.001	0.002	0.002	0.002	0.002	0.003
9.0				0.000	0.001	0.001	0.001	0.001	0.001
10.0					0.000	0.000	0.000	0.000	0.001

TABLE X

VALUES OF Θ FOR NONDIMENSIONAL TIME(τ) AND NONDIMENSIONAL LOCATION(R), ONE-DIMENSIONAL MODEL.

$\tau = 0.1 \rightarrow 0.9$
$R = 1.0 \rightarrow 10.0$
$\beta = 1.0$

τ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
R									
1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.2	0.583	0.662	0.695	0.713	0.724	0.732	0.737	0.741	0.744
1.4	0.301	0.417	0.471	0.501	0.521	0.534	0.544	0.551	0.556
1.6	0.135	0.248	0.309	0.346	0.370	0.388	0.400	0.409	0.416
1.8	0.052	0.138	0.195	0.233	0.259	0.278	0.292	0.303	0.311
2.0	0.017	0.071	0.118	0.153	0.178	0.197	0.211	0.222	0.231
2.2	0.005	0.034	0.068	0.097	0.120	0.137	0.151	0.162	0.170
2.4	0.001	0.015	0.038	0.059	0.078	0.094	0.106	0.116	0.125
2.6	0.000	0.006	0.020	0.035	0.050	0.063	0.074	0.083	0.090
2.8		0.002	0.010	0.020	0.031	0.041	0.050	0.058	0.065
3.0		0.001	0.005	0.011	0.019	0.026	0.034	0.040	0.046
3.2		0.000	0.002	0.006	0.011	0.018	0.022	0.027	0.032
3.4			0.001	0.003	0.006	0.010	0.014	0.018	0.022
3.6			0.000	0.001	0.003	0.006	0.009	0.012	0.015
3.8				0.001	0.002	0.003	0.005	0.008	0.010
4.0				0.000	0.001	0.002	0.003	0.005	0.007
5.0					0.000	0.000	0.000	0.000	0.001
6.0									0.000
7.0									
8.0									
9.0									
10.0									

TABLE XI

VALUES OF Θ FOR NONDIMENSIONAL TIME(τ) AND NONDIMENSIONAL LOCATION(R), ONE-DIMENSIONAL MODEL.

τ=1.0 9.0
R=1.0 10.0
β=1.0

τ / R	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0
1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.2	0.746	0.755	0.756	0.756	0.756	0.756	0.757	0.757	0.757
1.4	0.560	0.575	0.578	0.579	0.579	0.579	0.579	0.579	0.579
1.6	0.422	0.422	0.445	0.446	0.446	0.446	0.446	0.446	0.446
1.8	0.317	0.341	0.345	0.346	0.347	0.347	0.347	0.347	0.347
2.0	0.238	0.264	0.269	0.270	0.270	0.270	0.271	0.271	0.271
2.2	0.177	0.205	0.210	0.212	0.212	0.212	0.212	0.212	0.212
2.4	0.132	0.160	0.166	0.167	0.167	0.167	0.167	0.167	0.167
2.6	0.097	0.124	0.130	0.131	0.131	0.131	0.132	0.132	0.132
2.8	0.071	0.097	0.102	0.104	0.104	0.104	0.104	0.104	0.104
3.0	0.051	0.075	0.081	0.082	0.082	0.082	0.082	0.083	0.083
3.2	0.036	0.058	0.064	0.065	0.065	0.065	0.065	0.066	0.066
3.4	0.026	0.045	0.050	0.052	0.052	0.052	0.052	0.052	0.052
3.6	0.018	0.035	0.040	0.041	0.041	0.041	0.042	0.042	0.042
3.8	0.012	0.027	0.031	0.033	0.033	0.033	0.033	0.033	0.033
4.0	0.008	0.021	0.025	0.026	0.026	0.026	0.026	0.027	0.027
5.0	0.001	0.005	0.007	0.008	0.009	0.009	0.009	0.009	0.009
6.0	0.000	0.001	0.002	0.003	0.003	0.003	0.003	0.003	0.003
7.0		0.000	0.001	0.001	0.001	0.001	0.001	0.001	0.001
8.0			0.000	0.000	0.000	0.000	0.000	0.000	0.000
9.0									
10.0									

TABLE XII

VALUES OF Θ FOR NONDIMENSIONAL TIME(τ) AND NONDIMENSIONAL LOCATION(R), TWO-DIMENSIONAL MODEL(TRUMP SOLUTION)

τ=0.5→50.0
R=1.0→30.0
Θ=0.0
X=1R

τ / R	0.5	1.0	3.0	5.0	10.0	20.0	50.0	100.0
1.5	0.492	0.583	0.696	0.741	0.775	0.791	0.803	0.830
2.0	0.216	0.330	0.497	0.568	0.622	0.650	0.671	0.723
2.5	0.082	0.176	0.355	0.440	0.513	0.550	0.576	0.633
3.0	0.027	0.087	0.249	0.339	0.420	0.464	0.499	0.573
3.5	0.008	0.039	0.170	0.258	0.353	0.403	0.440	0.512
4.0	0.002	0.016	0.113	0.194	0.286	0.344	0.390	0.470
4.5	0.000	0.006	0.074	0.142	0.241	0.305	0.350	0.425
5.0		0.002	0.048	0.105	0.193	0.264	0.316	0.392
6.0		0.000	0.017	0.050	0.123	0.202	0.258	0.327
7.0			0.005	0.022	0.077	0.152	0.214	0.278
8.0			0.002	0.010	0.048	0.117	0.179	0.238
10.0			0.000	0.002	0.017	0.065	0.125	0.174
14.0				0.000	0.002	0.015	0.055	0.092
20.0					0.000	0.001	0.013	0.035
30.0						0.000	0.001	0.006

TABLE XIII

VALUES OF Θ FOR NONDIMENSIONAL TIME(τ) AND NONDIMENSIONAL LOCATION(R), TWO-DIMENSIONAL MODEL(TRUMP SOLUTION)

R=1.0 30.0
τ=0.5 50.0
Θ=0.0
X=5R

τ / R	0.5	1.0	3.0	5.0	10.0	20.0	50.0	100.0
1.5	0.491	0.582	0.695	0.739	0.770	0.786	0.796	0.819
2.0	0.216	0.330	0.496	0.566	0.617	0.643	0.661	0.699
2.5	0.083	0.176	0.354	0.438	0.506	0.539	0.563	0.611
3.0	0.028	0.087	0.247	0.337	0.413	0.453	0.486	0.541
3.5	0.008	0.040	0.170	0.256	0.343	0.388	0.424	0.485
4.0	0.002	0.016	0.112	0.192	0.280	0.332	0.374	0.437
4.5	0.000	0.008	0.073	0.072	0.141	0.232	0.290	0.332
5.0		0.003	0.047	0.103	0.188	0.251	0.297	0.361
6.0		0.000	0.017	0.049	0.119	0.188	0.239	0.300
7.0			0.005	0.021	0.073	0.141	0.196	0.253
8.0			0.002	0.009	0.045	0.107	0.163	0.217
10.0			0.000	0.002	0.015	0.057	0.112	0.158
14.0				0.000	0.001	0.013	0.047	0.084
20.0					0.000	0.001	0.011	0.032
30.0						0.000	0.001	0.006

TABLE XIV

VALUES OF Θ FOR NONDIMENSIONAL TIME(τ) AND NONDIMENSIONAL LOCATION(R), TWO-DIMENSIONAL MODEL(TRUMP SOLUTION)

τ=0.5 → 50.0
R=1.0 → 30.0
Θ=0.0
X=8.5R

τ / R	0.5	1.0	3.0	5.0	10.0	20.0	50.0	100.0
1.5	0.485	0.569	0.664	0.700	0.724	0.736	0.745	0.772
2.0	0.211	0.315	0.455	0.512	0.552	0.573	0.588	0.623
2.5	0.080	0.165	0.313	0.380	0.431	0.456	0.477	0.531
3.0	0.027	0.080	0.213	0.284	0.345	0.376	0.400	0.450
3.5	0.008	0.036	0.142	0.211	0.274	0.310	0.339	0.403
4.0	0.002	0.015	0.093	0.155	0.224	0.264	0.295	0.350
4.5	0.000	0.006	0.060	0.112	0.179	0.223	0.258	0.322
5.0		0.002	0.038	0.081	0.145	0.195	0.231	0.286
6.0		0.000	0.013	0.037	0.088	0.143	0.184	0.241
7.0			0.004	0.015	0.053	0.107	0.150	0.201
8.0			0.001	0.006	0.032	0.081	0.124	0.174
10.0			0.000	0.001	0.010	0.044	0.085	0.128
14.0				0.000	0.002	0.017	0.042	0.073
20.0					0.000	0.001	0.007	0.027
30.0						0.000	0.000	0.004

TABLE XV

VALUES OF Θ FOR NONDIMENSIONAL TIME(τ) AND NONDIMENSIONAL LOCATION(R), TWO-DIMENSIONAL MODEL(TRUMP SOLUTION)

τ=0.5→50.0
R=1.0→30.0
Θ=0.0
X=11.5R

τ / R	0.5	1.0	3.0	5.0	10.0	20.0	50.0	100.0
1.5	0.010	0.029	0.094	0.137	0.183	0.223	0.241	0.293
2.0	0.006	0.022	0.083	0.124	0.172	0.214	0.231	0.277
2.5	0.003	0.014	0.066	0.106	0.153	0.193	0.214	0.266
3.0	0.001	0.008	0.050	0.086	0.134	0.178	0.197	0.245
4.0		0.002	0.024	0.051	0.095	0.142	0.165	0.212
4.5		0.001	0.016	0.037	0.075	0.125	0.150	0.199
5.0		0.000	0.010	0.027	0.064	0.111	0.138	0.184
6.0			0.004	0.012	0.040	0.084	0.118	0.160
7.0			0.001	0.005	0.024	0.065	0.098	0.140
8.0			0.000	0.002	0.014	0.049	0.083	0.123
10.0				0.000	0.004	0.027	0.058	0.095
14.0					0.000	0.007	0.027	0.061
20.0						0.001	0.004	0.022
30.0						0.000	0.000	0.003

TABLE XVI

VALUES OF Θ FOR NONDIMENSIONAL TIME(τ) AND NONDIMENSIONAL LOCATION(R), TWO-DIMENSIONAL MODEL(TRUMP SOLUTION)

τ=0.5→50.0
R=1.0→30.0
Θ=0.0
X=13.5R

τ / R	0.5	1.0	3.0	5.0	10.0	20.0	50.0	100.0
1.5	0.000	0.001	0.013	0.030	0.060	0.102	0.118	0.166
2.0		0.000	0.012	0.027	0.058	0.101	0.116	0.156
2.5			0.010	0.024	0.053	0.095	0.112	0.158
3.0			0.008	0.020	0.048	0.090	0.108	0.147
3.5			0.006	0.016	0.042	0.083	0.102	0.146
4.0			0.004	0.012	0.036	0.077	0.097	0.136
4.5			0.003	0.009	0.030	0.069	0.092	0.133
5.0			0.002	0.006	0.025	0.063	0.087	0.124
6.0			0.001	0.003	0.016	0.050	0.076	0.113
7.0			0.000	0.001	0.010	0.039	0.067	0.101
8.0				0.001	0.006	0.031	0.058	0.091
10.0				0.000	0.002	0.017	0.042	0.073
14.0					0.001	0.004	0.018	0.041
20.0					0.000	0.003	0.004	0.020
30.0						0.000	0.000	0.003

TABLE XVII. EXPERIMENTAL DATA

RUN NO.	τ	r_p(cm)	r (cm)	T_o(°C)	T_c(°C)	T_p(°C)	Θ_c	ref_p (cm)	ref (cm)	R
1	1.0	0.72	0.46	22.6	29.9	48.5	0.282	7.62	4.77	2.23
10-21-71	2.0	0.85	0.53	22.6	29.9	48.5	0.282	7.62	4.77	2.57
(radial)	4.0	1.00	0.63	22.6	29.9	48.5	0.282	7.62	4.77	3.03
X=1R	6.0	1.16	0.72	22.6	29.9	48.5	0.282	7.62	4.77	3.49
	8.0	1.24	0.78	22.6	29.9	48.5	0.282	7.62	4.77	3.76
	10.0	1.32	0.83	22.6	29.9	48.5	0.282	7.62	4.77	3.99
2	0.5	0.71	0.45	23.9	29.9	48.5	0.244	7.32	4.57	2.14
11-5-71	1.5	0.81	0.51	23.9	29.9	48.5	0.244	7.32	4.57	2.45
(radial)	3.0	0.97	0.60	23.9	29.9	48.5	0.244	7.32	4.57	2.91
X=1R	4.0	1.07	0.67	23.9	29.9	48.5	0.244	7.32	4.57	3.20
	5.0	1.13	0.71	23.9	29.9	48.5	0.244	7.32	4.57	3.40
	6.0	1.21	0.75	23.9	29.9	48.5	0.244	7.32	4.57	3.63
	7.0	1.26	0.78	23.9	29.9	48.5	0.244	7.32	4.57	3.78
3	2.0	0.99	0.62	23.0	29.9	48.5	0.271	7.34	4.57	2.97
11-8-71	4.0	1.17	0.73	23.0	29.9	48.5	0.271	7.34	4.57	3.50
(radial)	6.0	1.28	0.80	23.0	29.9	48.5	0.271	7.34	4.57	3.85
X=1R	8.0	1.40	0.87	23.0	29.9	48.5	0.271	7.34	4.57	4.20
	10.0	1.50	0.93	23.0	29.9	48.5	0.271	7.34	4.57	4.49
4	2.0	0.99	0.62	23.0	29.9	48.5	0.271	7.32	4.57	2.97
11-9-71	4.0	1.17	0.73	23.0	29.9	48.5	0.271	7.32	4.57	3.50
(radial)	6.0	1.32	0.83	23.0	29.9	48.5	0.271	7.32	4.57	3.91
X=1R	8.0	1.41	0.88	23.0	29.9	48.5	0.271	7.32	4.57	4.23
	10.0	1.50	0.93	23.0	29.9	48.5	0.271	7.32	4.57	4.49
5	2.0	0.91	0.59	24.0	29.9	48.5	0.240	7.11	4.57	2.82
11-19-71	4.0	1.12	0.72	24.0	29.9	48.5	0.240	7.11	4.57	3.46
(radial)										
X=1R										

TABLE XVII. EXPERIMENTAL DATA (Cont'd)

RUN NO	τ	r_p(cm)	r(cm)	T_o(°C	T_c(°C)	T_D(°C)	Θ_c	ref_p (cm)	ref (cm)	R
6	2.0	0.91	0.59	23.6	29.9	48.5	0.253	7.11	4.57	2.83
11-20-71	6.0	1.28	0.83	23.6	29.9	48.5	0.253	7.11	4.57	3.98
(radial)	8.0	1.38	0.89	23.6	29.9	48.5	0.253	7.11	4.57	4.28
X=1R	10.0	1.48	0.95	23.6	29.9	48.5	0.253	7.11	4.57	4.56
7	2.0	0.86	0.58	24.2	29.9	48.5	0.235	6.83	4.57	2.79
11-21-71	4.0	1.09	0.73	24.2	29.9	48.5	0.235	6.83	4.57	3.51
(radial)	8.0	1.37	0.92	24.2	29.9	48.5	0.235	6.83	4.57	4.42
X=1R	10.0	1.42	0.95	24.2	29.9	48.5	0.235	6.83	4.57	4.59
8	1.0	0.61	0.41	24.2	29.9	48.5	0.235	6.83	4.57	1.97
11-21-71	3.0	0.99	0.66	24.2	29.9	48.5	0.235	6.83	4.57	3.20
(radial)	5.0	1.17	0.78	24.2	29.9	48.5	0.235	6.83	4.57	3.77
X=1R	7.0	1.32	0.88	24.2	29.9	48.5	0.235	6.83	4.57	4.26
	9.0	1.44	0.96	24.2	29.9	48.5	0.235	6.83	4.57	4.62
9	1.5	0.74	0.49	24.2	29.9	48.5	0.235	6.83	4.57	2.37
11-21-71	3.5	1.03	0.69	24.2	29.9	48.5	0.235	6.83	4.57	3.31
(radial)	5.5	1.18	0.79	24.2	29.9	48.5	0.235	6.83	4.57	3.82
X=1R	7.5	1.31	0.88	24.2	29.9	48.5	0.235	6.83	4.57	4.22
10	1.0	0.65	0.43	25.2	29.9	48.5	0.202	6.83	4.57	2.09
11-22-71	1.5	0.76	0.51	25.2	29.9	48.5	0.202	6.83	4.57	2.46
(radial)	2.0	0.84	0.56	25.2	29.9	48.5	0.202	6.83	4.57	2.70
X=1R	3.0	1.02	0.68	25.2	29.9	48.5	0.202	6.83	4.57	3.28
	3.5	1.14	0.76	25.2	29.9	48.5	0.202	6.83	4.57	3.68
	4.0	1.09	0.73	25.2	29.9	48.5	0.202	6.83	4.57	3.51
	5.0	1.26	0.84	25.2	29.9	48.5	0.202	6.83	4.57	4.05
	5.5	1.32	0.88	25.2	29.9	48.5	0.202	6.83	4.57	4.26
	6.0	1.24	0.83	25.2	29.9	48.5	0.202	6.83	4.57	4.01
	7.0	1.38	0.93	25.2	29.9	48.5	0.202	6.83	4.57	4.46

TABLE XVII. EXPERIMENTAL DATA (Cont'd)

RUN NO.	τ	r_p(cm)	r(cm)	T_o(°C)	T_t(°C)	T_p(°C)	Θ_c	ref_D (cm)	ref (cm)	R
10 (cont.)	7.5	1.47	0.99	25.2	29.9	48.5	0.202	6.83	4.57	4.75
	8.0	1.40	0.94	25.2	29.9	48.5	0.202	6.83	4.57	4.50
	9.0	1.52	1.02	25.2	29.9	48.5	0.202	6.83	4.57	4.91
	9.5	1.60	1.07	25.2	29.9	48.5	0.202	6.83	4.57	5.16
	10.0	1.52	1.02	25.2	29.9	48.5	0.202	6.83	4.57	4.91
11	2.0	0.71	0.46	22.5	29.9	48.5	0.285	3.40	2.21	2.23
11-28-71	4.0	0.91	0.59	22.5	29.9	48.5	0.285	3.40	2.21	2.86
(axial)	6.0	1.02	0.66	22.5	29.9	48.5	0.285	3.40	2.21	3.18
X = 1R	8.0	1.07	0.69	22.5	29.9	48.5	0.285	3.40	2.21	3.34
	10.0	1.12	0.73	22.5	29.9	48.5	0.285	3.40	2.21	3.50
X = 8.5R	2.0	0.61	0.40	22.5	29.9	48.5	0.285	3.40	2.21	1.91
	4.0	0.66	0.43	22.5	29.9	48.5	0.285	3.40	2.21	2.07
	6.0	0.81	0.53	22.5	29.9	48.5	0.285	3.40	2.21	2.54
	8.0	0.91	0.59	22.5	29.9	48.5	0.285	3.40	2.21	2.86
	10.0	0.97	0.63	22.5	29.9	48.5	0.285	3.40	2.21	3.02
X = 10R	2.0	0.51	0.33	22.5	29.9	48.5	0.285	3.40	2.21	1.59
	4.0	0.56	0.36	22.5	29.9	48.5	0.285	3.40	2.21	1.75
	6.0	0.66	0.43	22.5	29.9	48.5	0.285	3.40	2.21	2.07
	8.0	0.71	0.46	22.5	29.9	48.5	0.285	3.40	2.21	2.23
	10.0	0.76	0.49	22.5	29.9	48.5	0.285	3.40	22.1	2.39
X = 11.5R	2.0	0.33	0.20	22.5	29.9	48.5	0.285	3.40	22.1	1.00
	4.0	0.33	0.20	22.5	29.9	48.5	0.285	3.40	22.1	1.00
	6.0	0.33	0.20	22.5	29.9	48.5	0.285	3.40	22.1	1.00
	8.0	0.33	0.20	22.5	29.9	48.5	0.285	3.40	22.1	1.00
	10.0	0.41	0.26	22.5	29.9	48.5	0.285	3.40	22.1	1.27

```
      COMPUTER PROGRAM 1.

      EXTERNAL FUNC
      DIMENSION AYORR(31)
C     A=LOWER LIMIT OF INTEGRAL
      A=.001
C     B=UPPER LIMIT OF INTEGRAL
      B=10.0
C     N=NUMBER OF INTERVALS
      N=290
C     BETA=BLOOD FLOW TERM
      BETA=0.0
C     INPUT Y BESSEL VALUES FOR A AND AR
      AYORR(1)=-4.471416
      AYORR(2)=-4.410739
      AYORR(3)=-4.355347
      AYORR(4)=-4.304389
      AYORR(5)=-4.257210
      AYORR(6)=-4.213287
      AYORR(7)=-4.172200
      AYORR(8)=-4.133606
      AYORR(9)=-4.097217
      AYORR(10)=-4.062797
      AYORR(11)=-4.030141
      AYORR(12)=-3.99080
      AYORR(13)=-3.969463
      AYORR(14)=-3.941165
      AYORR(15)=-3.914070
      AYORR(16)=-3.888082
      AYORR(17)=-3.863112
      AYORR(18)=-3.839086
      AYORR(19)=-3.815933
      AYORR(20)=-3.793592
      AYORR(21)=-3.772010
      AYORR(22)=-3.751135
      AYORR(23)=-3.730922
      AYORR(24)=-3.711331
      AYORR(25)=-3.692327
      AYORR(26)=-3.673872
      AYORR(27)=-3.655937
      AYORR(28)=-3.638493
      AYORR(29)=-3.621515
      AYORR(30)=-3.604978
      AYORR(31)=-3.588861
      TSUM=0.0
      RSUM=0.0
START OUTER DO LOOP.  TIME VARIES FROM 0.1 TO 1.0
      DO 1000 I=1,10
```

```
C       START INNER DO LOOP.  RADIUS VARIES FROM 1.0 TO 4.0 FOR EACH
  VALUE OF TIME.
        DO 1100 K=1,31
        R=RSUM+1.0
        T=TSUM+0.1
        AYO=-4.4714155
        AYOR=AYORR(K)
C       EVALUATE INTEGRAL USING SIMPSON,S RULE.
        SINT=SIMP(FUNC,A,B,N,T,R,BETA,AYO,AYOR)
C       COMBINE TERMS TO GET ANSWER.
        PI=3.14159265
        ANS=1.0-(2/PI)*(SINT)
        WRITE(6,2000) T,R,ANS
 2000 FORMAT(' ','T=',F5.1,5X,'R=',F4.1,5X,'ANS=',F5.3)
C       RECALCULATE USING NEXT HIGHER VALUE OF R.
 1100 RSUM=RSUM+0.1
C       RECALCULATE FOR NEXT VALUE OF T.
 1000 TSUM=TSUM+0.1
        STOP
        END
C       FUNCTION FUNC SUBPROGRAM

        FUNCTION FUNC(U,T,RJOU,YOU,RJOUR,YOUR,RJOUS,YOUS,BETA)
        FUNC=EXP(-T*U**2)*((RJOU*YOUR)-(RJOUR*YOU))/((U+BETA/U)*(YOUS+RJOU
      2S))
        RETURN
        END
C       SIMPSON,S RULE SUBPROGRAM

        FUNCTION SIMP(FUNC,A,B,N,T,R,BETA,AYO,AYOR)
C       FUNC=FUNCTION NAME
C       A=INITIAL VALUE OF U
C       B=FINAL VALUE OF U
C       N=NUMBER OR INTERVALS,EVEN.
C       EVALUATE INTEGRAL OF FUNC FROM U=A TO U=B USING N INTERVALS.
 9005 H=(B-A)/FLOAT(N)
C       D IS ACCURACY OF J BESSEL FUNCTION DESIRED.
        D=.00001
C       CALCULATE BESSEL FUNCTIONS FOR LOWER LIMIT OF INTEGRAL, A=.001
        CALL BESJ(A,0,AJO,D,IER)
        AYOS=(AYO)**2
        AJOS=(AJO)**2
 9001 AR=A*R
        CALL BESJ(AR,0,AJOR,D,IER)
C       CALCULATE BESSEL FUNCTIONS OF UPPER LIMIT OF INTEGRAL, B.
        CALL BESJ(B,0,BJO,D,IER)
```

```
 9003 CALL BESY(B,0,BY0,IER)
      BJOS=(BJ0)**2
      BYOS=(BY0)**2
      BR=B*R
      CALL BESJ(BR,0,BJOR,D,IER)
 9004 CALL BESY(BR,0,BYOR,IER)
      EVNSUM=0.0
      ODDSUM=0.0
      ENDSUM=FUNC(A,T,AJ0,AY0,AJOR,AYOR,AJOS,AYOS,BETA)+FUNC(B,T,BJ0,BY0
     2,BJOR,BYOR,BJOS,BYOS,BETA)
      M=N-1
      J=1
      DO 1500 I=1,M
      X=A+H*FLOAT(I)
C     CALCULATE BESSEL FUNCTIONS FOR X.
      CALL BESJ(X,0,XJ0,D,IER)
      CALL BESY(X,0,XY0,IER)
      XJOS=(XJ0)**2
      XYOS=(XY0)**2
      XR=X*R
      CALL BESJ(XR,0,XJOR,D,IER)
      CALL BESY(XR,0,XYOR,IER)
      GO TO (1001,1002),J
 1001 ODDSUM=ODDSUM+FUNC(X,T,XJ0,XY0,XJOR,XYOR,XJOS,XYOS,BETA)
      J=2
      GO TO 1500
 1002 EVNSUM=EVNSUM+FUNC(X,T,XJ0,XY0,XJOR,XYOR,XJOS,XYOS,BETA)
      J=1
 1500 CONTINUE
 1600 SIMP=(H/3.)*(ENDSUM+4.0*ODDSUM+2.0*EVNSUM)
      RETURN
      END
C                                                                      BESK  10
C     ..................................................................BESK  20
C                                                                      BESK  30
C        SUBROUTINE BESK                                               BESK  40
C                                                                      BESK  50
C           COMPUTE THE K BESSEL FUNCTION FOR A GIVEN ARGUMENT AND ORDERBESK  60
C                                                                      BESK  70
C        USAGE                                                         BESK  80
C           CALL BESK(X,N,BK,IER)                                      BESK  90
C                                                                      BESK 100
C        DESCRIPTION OF PARAMETERS                                     BESK 110
C           X  -THE ARGUMENT OF THE K BESSEL FUNCTION DESIRED          BESK 120
C           N  -THE ORDER OF THE K BESSEL FUNCTION DESIRED             BESK 130
C           BK -THE RESULTANT K BESSEL FUNCTION                        BESK 140
C           IER-RESULTANT ERROR CODE WHERE                             BESK 150
C              IER=0  NO ERROR                                         BESK 160
```

```
C                IER=1   N IS NEGATIVE                                          BESK 170
C                IER=2   X IS ZERO OR NEGATIVE                                 BESK 180
C                IER=3   X .GT. 170, MACHINE RANGE EXCEEDED                    BESK 190
C                IER=4   BK .GT. 10**70                                        BESK 200
C                                                                              BESK 210
C          REMARKS                                                             BESK 220
C             N MUST BE GREATER THAN OR EQUAL TO ZERO                          BESK 230
C                                                                              BESK 240
C          SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                       BESK 250
C             NONE                                                             BESK 260
C                                                                              BESK 270
C          METHOD                                                              BESK 280
C             COMPUTES ZERO ORDER AND FIRST ORDER BESSEL FUNCTIONS USING       BESK 290
C             SERIES APPROXIMATIONS AND THEN COMPUTES N TH ORDER FUNCTION      BESK 300
C             USING RECURRENCE RELATION.                                       BESK 310
C             RECURRENCE RELATION AND POLYNOMIAL APPROXIMATION TECHNIQUE       BESK 320
C             AS DESCRIBED BY A.J.M.HITCHCOCK,'POLYNOMIAL APPROXIMATIONS       BESK 330
C             TO BESSEL FUNCTIONS OF ORDER ZERO AND ONE AND TO RELATED         BESK 340
C             FUNCTIONS', M.T.A.C., V.11,1957,PP.86-88, AND G.N. WATSON,       BESK 350
C             'A TREATISE ON THE THEORY OF BESSEL FUNCTIONS', CAMBRIDGE        BESK 360
C             UNIVERSITY PRESS, 1958, P. 62                                    BESK 370
C                                                                              BESK 380
C     ..........................................................               BESK 390
C                                                                              BESK 400

      SUBROUTINE BESK(X,N,BK,IER)                                              BESK 41
      DIMENSION T(12)                                                          BESK 420
      BK=.0                                                                    BESK 430
      IF(N)10,11,11                                                            BESK 440
   10 IER=1                                                                    BESK 450
      RETURN                                                                   BESK 460
   11 IF(X)12,12,20                                                            BESK 470
   12 IER=2                                                                    BESK 480
      RETURN                                                                   BESK 490
   20 IF(X-170.0)22,22,21                                                      BESK 500
   21 IER=3                                                                    BESK 510
      RETURN                                                                   BESK 520
   22 IER=0                                                                    BESK 530
      IF(X-1.)36,36,25                                                         BESK 540
   25 A=EXP(-X)                                                                BESK 550
      B=1./X                                                                   BESK 560
      C=SQRT(B)                                                                BESK 570
      T(1)=B                                                                   BESK 580
      DO 26 L=2,12                                                             BESK 590
   26 T(L)=T(L-1)*B                                                            BESK 600
      IF(N-1)27,29,27                                                          BESK 610
C                                                                              BESK 620
```

```
C       COMPUTE KO USING POLYNOMIAL APPROXIMATION                       BESK 630
C                                                                       BESK 640
   27 GO=A*(1.2533141-.1566642*T(1)+.08811128*T(2)-.09139095*T(3)       BESK 650
     2+.1344596*T(4)-.2299850*T(5)+.3792410*T(6)-.5247277*T(7)          BESK 660
     3+.5575368*T(8)-.4262633*T(9)+.2184518*T(10)-.06680977*T(11)       BESK 670
     4+.00918938 3*T(12))*C                                             BESK 680
      IF(N)20,28,29                                                     BESK 690
   28 BK=GO                                                             BESK 700
      RETURN                                                            BESK 710
C                                                                       BESK 720
C       COMPUTE K1 USING POLYNOMIAL APPROXIMATION                       BESK 730
C                                                                       BESK 740
   29 G1=A*(1.2533141+.4699927*T(1)-.1468583*T(2)+.1280427*T(3)         BESK 750
     2-.1736432*T(4)+.2847618*T(5)-.4594342*T(6)+.6283381*T(7)          BESK 760
     3-.6632295*T(8)+.5050239*T(9)-.2581304*T(10)+.07880001*T(11)       BESK 770
     4-.01082418*T(12))*C                                               BESK 780
      IF(N-1)20,30,31                                                   BESK 790
   30 BK=G1                                                             BESK 800
      RETURN                                                            BESK 810
C                                                                       BESK 820
C       FROM KO,K1 COMPUTE KN USING RECURRENCE RELATION                 BESK 830
C                                                                       BESK 840
   31 DO 35 J=2,N                                                       BESK 850
      GJ=2.*(FLOAT(J)-1.)*G1/X+GO                                       BESK 860
      IF(GJ-1.0E70)33,33,32                                             BESK 870
   32 IER=4                                                             BESK 880
      GO TO 34                                                          BESK 890
   33 GO=G1                                                             BESK 900
   35 G1=GJ                                                             BESK 910
   34 BK=GJ                                                             BESK 920
      RETURN                                                            BESK 930
   36 B=X/2.                                                            BESK 940
      A=.5772157+ALOG(B)                                                BESK 950
      C=B*B                                                             BESK 960
      IF(N-1)37,43,37                                                   BESK 970
C                                                                       BESK 980
C       COMPUTE KO USING SERIES EXPANSION                               BESK 990
C                                                                       BESK1000
   37 GO=-A                                                             BESK1010
      X2J=1.                                                            BESK1020
      FACT=1.                                                           BESK1030
      HJ=.0                                                             BESK1040
      DO 40 J=1,6                                                       BESK1050
      RJ=1./FLOAT(J)                                                    BESK1060
      X2J=X2J*C                                                         BESK1070
      FACT=FACT*RJ*RJ                                                   BESK1080
      HJ=HJ+RJ                                                          BESK1090
   40 GO=GO+X2J*FACT*(HJ-A)                                             BESK1100
```

```
      IF(N)43,42,43                                                          BESK1110
   42 BK=G0                                                                  BESK1120
      RETURN                                                                 BESK1130
C                                                                            BESK1140
C     COMPUTE K1 USING SERIES EXPANSION                                      BESK1150
C                                                                            BESK1160
   43 X2J=B                                                                  BESK1170
      FACT=1.                                                                BESK1180
      HJ=1.                                                                  BESK1190
      G1=1./X+X2J*(.5+A-HJ)                                                  BESK1200
      DO 50 J=2,8                                                            BESK1210
      X2J=X2J*C                                                              BESK1220
      RJ=1./FLOAT(J)                                                         BESK1230
      FACT=FACT*RJ*RJ                                                        BESK1240
      HJ=HJ+RJ                                                               BESK1250
   50 G1=G1+X2J*FACT*(.5+(A-HJ)*FLOAT(J))                                    BESK1260
      IF(N-1)31,52,31                                                        BESK1270
   52 BK=G1                                                                  BESK1280
      RETURN                                                                 BESK1290
      END                                                                    BESK1300
C                                                                            BESJ  10
CC    ..........................................................................BESJ  20
CC                                                                           BESJ  30
CC       SUBROUTINE BESJ                                                     BESJ  40
CC                                                                           BESJ  50
CC       PURPOSE                                                             BESJ  60
CC          COMPUTE THE J BESSEL FUNCTION FOR A GIVEN ARGUMENT AND ORDER     BESJ  70
CC                                                                           BESJ  80
CC       USAGE                                                               BESJ  90
CC          CALL BESJ(X,N,BJ,D,IER)                                          BESJ 100
CC                                                                           BESJ 110
CC       DESCRIPTION OF PARAMETERS                                           BESJ 120
CC          X  -THE ARGUMENT OF THE J BESSEL FUNCTION DESIRED                BESJ 130
CC          N  -THE ORDER OF THE J BESSEL FUNCTION DESIRED                   BESJ 140
CC          BJ -THE RESULTANT J BESSEL FUNCTION                              BESJ 150
CC          D  -REQUIRED ACCURACY                                            BESJ 160
CC          IER-RESULTANT ERROR CODE WHERE                                   BESJ 170
CC             IER=0  NO ERROR                                               BESJ 180
CC             IER=1  N IS NEGATIVE                                          BESJ 190
CC             IER=2  X IS NEGATIVE OR ZERO                                  BESJ 200
CC             IER=3  REQUIRED ACCURACY NOT OBTAINED                         BESJ 210
CC             IER=4  RANGE OF N COMPARED TO X NOT CORRECT (SEE REMARKS)     BESJ 220
CC                                                                           BESJ 230
CC       REMARKS                                                             BESJ 240
CC          N MUST BE GREATER THAN OR EQUAL TO ZERO, BUT IT MUST BE          BESJ 250
CC          LESS THAN                                                        BESJ 260
CC             20+10*X-X** 2/3   FOR X LESS THAN OR EQUAL TO 15             BESJ 270
CC             90+X/2            FOR X GREATER THAN 15                      BESJ 280
```

```
C                                                                       BESJ 290
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                  BESJ 300
C           NONE                                                        BESJ 310
C                                                                       BESJ 320
C        METHOD                                                         BESJ 330
C           RECURRENCE RELATION TECHNIQUE DESCRIBED BY H. GOLDSTEIN AND BESJ 340
C           R.M. THALER,'RECURRENCE TECHNIQUES FOR THE CALCULATION OF   BESJ 350
C           BESSEL FUNCTIONS',M.T.A.C.,V.13,PP.102-108 AND I.A. STEGUN  BESJ 360
C           AND M. ABRAMOWITZ,'GENERATION OF BESSEL FUNCTIONS ON HIGH   BESJ 370
C           SPEED COMPUTERS',M.T.A.C.,V.11,1957,PP.255-257              BESJ 380
C                                                                       BESJ 390
C     .....BESJ 400
C                                                                       BESJ 410

      SUBROUTINE BESJ(X,N,BJ,D,IER)                                     BESJ 420
C                                                                       BESJ 430
      BJ=.0                                                             BESJ 440
      IF(N)10,20,20                                                     BESJ 450
   10 IER=1                                                             BESJ 460
      RETURN                                                            BESJ 470
   20 IF(X)30,30,31                                                     BESJ 480
   30 IER=2                                                             BESJ 490
      RETURN                                                            BESJ 500
   31 IF(X-15.)32,32,34                                                 BESJ 510
   32 NTEST=20.+10.*X-X** 2/3                                           BESJ 520
      GO TO 36                                                          BESJ 530
   34 NTEST=90.+X/2.                                                    BESJ 540
   36 IF(N-NTEST)40,38,38                                               BESJ 550
   38 IER=4                                                             BESJ 560
      RETURN                                                            BESJ 570
   40 IER=0                                                             BESJ 580
      N1=N+1                                                            BESJ 590
      BPREV=.0                                                          BESJ 600
C                                                                       BESJ 610
C     COMPUTE STARTING VALUE OF M                                       BESJ 620
C                                                                       BESJ 630
      IF(X-5.)50,60,60                                                  BESJ 640
   50 MA=X+6.                                                           BESJ 650
      GO TO 70                                                          BESJ 660
   60 MA=1.4*X+60./X                                                    BESJ 670
   70 MB=N+IFIX(X)/4+2                                                  BESJ 680
      MZERO=MAXO(MA,MB)                                                 BESJ 690
C                                                                       BESJ 700
C     SET UPPER LIMIT OF M                                              BESJ 710
C                                                                       BESJ 720
      MMAX=NTEST                                                        BESJ 730
  100 DO 190 M=MZERO,MMAX,3                                             BESJ 740
```

```
C                                                                       BESJ 750
C     SET F(M),F(M-1)                                                   BESJ 760
C                                                                       BESJ 770
      FM1=1.0E-28                                                       BESJ 780
      FM=.0                                                             BESJ 790
      ALPHA=.0                                                          BESJ 800
      IF(M-(M/2)*2)120,110,120                                          BESJ 810
  110 JT=-1                                                             BESJ 820
      GO TO 130                                                         BESJ 830
  120 JT=1                                                              BESJ 840
  130 M2=M-2                                                            BESJ 850
      DO 160 K=1,M2                                                     BESJ 860
      MK=M-K                                                            BESJ 870
      BMK=2.*FLOAT(MK)*FM1/X-FM                                         BESJ 880
      FM=FM1                                                            BESJ 890
      FM1=BMK                                                           BESJ 900
      IF(MK-N-1)150,140,150                                             BESJ 910
  140 BJ=BMK                                                            BESJ 920
  150 JT=-JT                                                            BESJ 930
      S=1+JT                                                            BESJ 940
  160 ALPHA=ALPHA+BMK*S                                                 BESJ 950
      BMK=2.*FM1/X-FM                                                   BESJ 960
      IF(N)180,170,180                                                  BESJ 970
  170 BJ=BMK                                                            BESJ 980
  180 ALPHA=ALPHA+BMK                                                   BESJ 990
      BJ=BJ/ALPHA                                                       BESJ1000
      IF(ABS(BJ-BPREV)-ABS(D*BJ))200,200,190                            BESJ1010
  190 BPREV=BJ                                                          BESJ1020
      IER=3                                                             BESJ1030
  200 RETURN                                                            BESJ1040
      END                                                               BESJ1050
C                                                                       BESY  10
C     .................................................................BESY  20
C                                                                       BESY  30
C        SUBROUTINE BESY                                                BESY  40
C                                                                       BESY  50
C        PURPOSE                                                        BESY  60
C           COMPUTE THE Y BESSEL FUNCTION FOR A GIVEN ARGUMENT AND ORDERBESY  70
C                                                                       BESY  80
C        USAGE                                                          BESY  90
C           CALL BESY(X,N,BY,IER)                                       BESY 100
C                                                                       BESY 110
C        DESCRIPTION OF PARAMETERS                                      BESY 120
C           X  -THE ARGUMENT OF THE Y BESSEL FUNCTION DESIRED           BESY 130
C           N  -THE ORDER OF THE Y BESSEL FUNCTION DESIRED              BESY 140
C           BY -THE RESULTANT Y BESSEL FUNCTION                         BESY 150
C           IER-RESULTANT ERROR CODE WHERE                              BESY 160
C              IER=0   NO ERROR                                         BESY 170
```

```
C                IER=1   N IS NEGATIVE                                          BESY 180
C                IER=2   X IS NEGATIVE OR ZERO                                 BESY 190
C                IER=3   BY HAS EXCEEDED MAGNITUDE OF 10**70                   BESY 200
C                                                                              BESY 210
C          REMARKS                                                             BESY 220
C             VERY SMALL VALUES OF X MAY CAUSE THE RANGE OF THE LIBRARY        BESY 230
C             FUNCTION ALOG TO BE EXCEEDED                                     BESY 240
C             X MUST BE GREATER THAN ZERO                                      BESY 250
C             N MUST BE GREATER THAN OR EQUAL TO ZERO                          BESY 260
C                                                                              BESY 270
C          SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                       BESY 280
C             NONE                                                             BESY 290
C                                                                              BESY 300
C          METHOD                                                              BESY 310
C             RECURRENCE RELATION AND POLYNOMIAL APPROXIMATION TECHNIQUE       BESY 320
C             AS DESCRIBED BY A.J.M.HITCHCOCK,'POLYNOMIAL APPROXIMATIONS       BESY 330
C             TO BESSEL FUNCTIONS OF ORDER ZERO AND ONE AND TO RELATED         BESY 340
C             FUNCTIONS', M.T.A.C., V.11,1957,PP.86-88, AND G.N. WATSON,       BESY 350
C             'A TREATISE ON THE THEORY OF BESSEL FUNCTIONS', CAMBRIDGE        BESY 360
C             UNIVERSITY PRESS, 1958, P. 62                                    BESY 370
C                                                                              BESY 380
C     ..........................................................................BESY 390
C                                                                              BESY 400


      SUBROUTINE BESY(X,N,BY,IER)                                              BESY 410
C                                                                              BESY 420
CC    CHECK FOR ERRORS IN N AND X                                              BESY 430
C                                                                              BESY 440
      IF(N)180,10,10                                                           BESY 450
   10 IER=0                                                                    BESY 460
      IF(X)190,190,20                                                          BESY 470
C                                                                              BESY 480
CC    BRANCH IF X LESS THAN OR EQUAL 4                                         BESY 490
C                                                                              BESY 500
   20 IF(X-4.0)40,40,30                                                        BESY 510
C                                                                              BESY 520
CC      COMPUTE Y0 AND Y1 FOR X GREATER THAN 4                                 BESY 530
C                                                                              BESY 540
   30 T1=4.0/X                                                                 BESY 550
      T2=T1*T1                                                                 BESY 560
      P0=((((-.0000037043*T2+.0000173565)*T2-.0000487613)*T2                   BESY 570
     1  +.00017343)*T2-.001753062)*T2+.3989423                                 BESY 580
      Q0=((((.0000032312*T2-.0000142078)*T2+.0000342468)*T2                    BESY 590
     1  -.0000869791)*T2+.0004564324)*T2-.01246694                             BESY 600
      P1=((((.0000042414*T2-.0000200920)*T2+.0000580759)*T2                    BESY 610
     1  -.000223203)*T2+.002921826)*T2+.3989423                                BESY 620
      Q1=((((-.0000036594*T2+.00001622)*T2-.0000398708)*T2                     BESY 630
```

```
     1  +.0001064741)*T2-.0006390400)*T2+.03740084                        BESY 640
        A=2.0/SQRT(X)                                                     BESY 650
        B=A*T1                                                            BESY 660
        C=X-.7853982                                                      BESY 670
        Y0=A*P0*SIN(C)+B*Q0*COS(C)                                        BESY 680
        Y1=-A*P1*COS(C)+B*Q1*SIN(C)                                       BESY 690
        GO TO 90                                                          BESY 700
C                                                                         BESY 710
CC        COMPUTE Y0 AND Y1 FOR X LESS THAN OR EQUAL TO 4                 BESY 720
C                                                                         BESY 730
   40   XX=X/2.                                                           BESY 740
        X2=XX*XX                                                          BESY 750
        T=ALOG(XX)+.5772157                                               BESY 760
        SUM=0.                                                            BESY 770
        TERM=T                                                            BESY 780
        Y0=T                                                              BESY 790
        DO 70 L=1,15                                                      BESY 800
        IF(L-1)50,60,50                                                   BESY 810
   50   SUM=SUM+1./FLOAT(L-1)                                             BESY 820
   60   FL=L                                                              BESY 830
        TS=T-SUM                                                          BESY 840
        TERM=(TERM*(-X2)/FL**2)*(1.-1./(FL*TS))                           BESY 850
   70   Y0=Y0+TERM                                                        BESY 860
        TERM = XX*(T-.5)                                                  BESY 870
        SUM=0.                                                            BESY 880
        Y1=TERM                                                           BESY 890
        DO 80 L=2,16                                                      BESY 900
        SUM=SUM+1./FLOAT(L-1)                                             BESY 910
        FL=L                                                              BESY 920
        FL1=FL-1.                                                         BESY 930
        TS=T-SUM                                                          BESY 940
        TERM=(TERM*(-X2)/(FL1*FL))*((TS-.5/FL)/(TS+.5/FL1))               BESY 950
   80   Y1=Y1+TERM                                                        BESY 960
        PI2=.6366198                                                      BESY 970
        Y0=PI2*Y0                                                         BESY 980
        Y1=-PI2/X+PI2*Y1                                                  BESY 990
C                                                                         BESY1000
CC      CHECK IF ONLY Y0 OR Y1 IS DESIRED                                 BESY1010
C                                                                         BESY1020
   90   IF(N-1)100,100,130                                                BESY1030
C                                                                         BESY1040
CC      RETURN EITHER Y0 OR Y1 AS REQUIRED                                BESY1050
C                                                                         BESY1060
  100   IF(N)110,120,110                                                  BESY1070
  110   BY=Y1                                                             BESY1080
        GO TO 170                                                         BESY1090
  120   BY=Y0                                                             BESY1100
        GO TO 170                                                         BESY1110
```

```
C                                                                       BESY1120
C     PERFORM RECURRENCE OPERATIONS TO FIND YN(X)                        BESY1130
C                                                                       BESY1140
  130 YA=Y0                                                             BESY1150
      YB=Y1                                                             BESY1160
      K=1                                                               BESY1170
  140 T=FLOAT(2*K)/X                                                    BESY1180
      YC=T*YB-YA                                                        BESY1190
      IF(ABS(YC)-1.0E70)145,145,141                                     BESY1200
  141 IER=3                                                             BESY1210
      RETURN                                                            BESY1220
  145 K=K+1                                                             BESY1230
      IF(K-N)150,160,150                                                BESY1240
  150 YA=YB                                                             BESY1250
      YB=YC                                                             BESY1260
      GO TO 140                                                         BESY1270
  160 BY=YC                                                             BESY1280
  170 RETURN                                                            BESY1290
  180 IER=1                                                             BESY1300
      RETURN                                                            BESY1310
  190 IER=2                                                             BESY1320
      RETURN                                                            BESY1330
      END                                                               BESY1340
```

```
      COMPUTER PROGRAM 2.

      EXTERNAL FUNC
      DIMENSION AYORR(5)
C     A=LOWER LIMIT OF INTEGRAL
      A=.001
C     B=UPPER LIMIT OF INTEGRAL
      B=10.0
C     N=NUMBER OF INTERVALS
      N=290
C     BETA=BLOOD FLOW TERM
      BETA=0.0
C     INPUT Y BESSEL VALUES FOR A AND AR
      DIMENSION AYORR(7)
      AYORR(1)=-3.588861
      AYORR(2)=-3.446794
      AYORR(3)=-3.330711
      AYORR(4)=-3.232564
      AYORR(5)=-3.147543
      AYORR(6)=-3.072545
      AYORR(7)=-3.005455
C     START OUTER DO LOOP.  TIME VARIES FROM 0.1 TO 1.0
      TSUM=0.0
      DO 1000 I=1,10
      RSUM=0.0
C     START INNER DO LOOP.  RADIUS VARIES FROM 4.0 TO10.0 FOR EACH
C     VALUE OF TIME.
      DO 1100 K=1,7
      R=RSUM+4.0
      T=TSUM+0.1
      AY0=-4.4714155
      AYOR=AYORR(K)
C     EVALUATE INTEGRAL USING SIMPSON,S RULE.
      SINT=SIMP(FUNC,A,B,N,T,R,BETA,AY0,AYOR)
C     COMBINE TERMS TO GET ANSWER.
      PI=3.14159265
      ANS=1.0-(2/PI)*(SINT)
      WRITE(6,2000) T,R,ANS
 2000 FORMAT(' ','T=',F5.1,5X,'R=',F4.1,5X,'ANS=',F5.3)
C     RECALCULATE USING NEXT HIGHER VALUE OF R.
 1100 RSUM=RSUM+1.0
C     RECALCULATE FOR NEXT VALUE OF T.
 1000 TSUM=TSUM+0.1
      STOP
      END
C     FUNCTION FUNC SUBPROGRAM
```

```
      FUNCTION FUNC(U,T,RJOU,YOU,RJOUR,YOUR,RJOUS,YOUS,BETA)
      FUNC=EXP(-T*U**2)*((RJOU*YOUR)-(RJOUR*YOU))/((U+BETA/U)*(YOUS+RJOU
     2S))
      RETURN
      END
C     SIMPSON,S RULE SUBPROGRAM

      FUNCTION SIMP(FUNC,A,B,N,T,R,BETA,AYO,AYOR)
C     FUNC=FUNCTION NAME
C     A=INITIAL VALUE OF U
C     B=FINAL VALUE OF U
C     N=NUMBER OR INTERVALS,EVEN.
C     EVALUATE INTEGRAL OF FUNC FROM U=A TO U=B USING N INTERVALS.
 9005 H=(B-A)/FLOAT(N)
C     D IS ACCURACY OF J BESSEL FUNCTION DESIRED.
      D=.00001
C     CALCULATE BESSEL FUNCTIONS FOR LOWER LIMIT OF INTEGRAL, A=.001
      CALL BESJ(A,0,AJO,D,IER)
      AYOS=(AYO)**2
      AJOS=(AJO)**2
 9001 AR=A*R
      CALL BESJ(AR,0,AJOR,D,IER)
C     CALCULATE BESSEL FUNCTIONS OF UPPER LIMIT OF INTEGRAL, B.
      CALL BESJ(B,0,BJO,D,IER)
 9003 CALL BESY(B,0,BYO,IER)
      BJOS=(BJO)**2
      BYOS=(BYO)**2
      BR=B*R
      CALL BESJ(BR,0,BJOR,D,IER)
 9004 CALL BESY(BR,0,BYOR,IER)
      EVNSUM=0.0
      ODDSUM=0.0
      ENDSUM=FUNC(A,T,AJO,AYO,AJOR,AYOR,AJOS,AYOS,BETA)+FUNC(B,T,BJO,BYO
     2,BJOR,BYOR,BJOS,BYOS,BETA)
      M=N-1
      J=1
      DO 1500 I=1,M
      X=A+H*FLOAT(I)
C     CALCULATE BESSEL FUNCTIONS FOR X.
      CALL BESJ(X,0,XJO,D,IER)
      CALL BESY(X,0,XYO,IER)
      XJOS=(XJO)**2
      XYOS=(XYO)**2
      XR=X*R
      CALL BESJ(XR,0,XJOR,D,IER)
      CALL BESY(XR,0,XYOR,IER)
      GO TO (1001,1002),J
```

```
1001  ODDSUM=ODDSUM+FUNC(X,T,XJ0,XY0,XJ0R,XY0R,XJ0S,XY0S,BETA)
      J=2
      GO TO 1500
1002  EVNSUM=EVNSUM+FUNC(X,T,XJ0,XY0,XJ0R,XY0R,XJ0S,XY0S,BETA)
      J=1
1500  CONTINUE
1600  SIMP=(H/3.)*(ENDSUM+4.0*ODDSUM+2.0*EVNSUM)
      RETURN
      END
```

COMPUTER PROGRAM 3.

```
      EXTERNAL FUNC
      DIMENSION AYORR(5)
C     A=LOWER LIMIT OF INTEGRAL
      A=.001
C     B=UPPER LIMIT OF INTEGRAL
      B=10.0
C     N=NUMBER OF INTERVALS
      N=290
C     BETA=BLOOD FLOW TERM
      BETA=0.0
C     INPUT Y BESSEL VALUES FOR A AND AR
      AYORR(1)=-3.005456
      AYORR(2)=-2.563955
      AYORR(3)=-2.305486
      AYORR(4)=-2.121901
      AYORR(5)=-1.979311
C     START OUTER DO LOOP.  TIME VARIES FROM 0.1 TO 1.0
      TSUM=0.0
      DO 1000 I=1,10
      RSUM=0.0
C     START INNER DO LOOP.  RADIUS VARIES FROM 10.0 TO 50.0 FOR EACH
C     VALUE OF TIME.
      DO 1100 K=1,5
      R=RSUM+10.0
      T=TSUM+0.1
      AYO=-4.4714155
      AYOR=AYORR(K)
C     EVALUATE INTEGRAL USING SIMPSON,S RULE.
      SINT=SIMP(FUNC,A,B,N,T,R,BETA,AYO,AYOR)
C     COMBINE TERMS TO GET ANSWER.
      PI=3.14159265
      ANS=1.0-(2/PI)*(SINT)
      WRITE(6,2000) T,R,ANS
 2000 FORMAT(' ','T=',F5.1,5X,'R=',F4.1,5X,'ANS=',F5.3)
C     RECALCULATE USING NEXT HIGHER VALUE OF R.
 1100 RSUM=RSUM+10.0
C     RECALCULATE FOR NEXT VALUE OF T.
 1000 TSUM=TSUM+0.1
      STOP
      END
C     FUNCTION FUNC SUBPROGRAM
```

```
      FUNCTION FUNC(U,T,RJOU,YOU,RJOUR,YOUR,RJOUS,YOUS,BETA)
      FUNC=EXP(-T*U**2)*((RJOU*YOUR)-(RJOUR*YOU))/((U+BETA/U)*(YOUS+RJOU
     2S))
      RETURN
      END
C     SIMPSON,S RULE SUBPROGRAM

      FUNCTION SIMP(FUNC,A,B,N,T,R,BETA,AYO,AYOR)
C     FUNC=FUNCTION NAME
C     A=INITIAL VALUE OF U
C     B=FINAL VALUE OF U
C     N=NUMBER OR INTERVALS,EVEN.
C     EVALUATE INTEGRAL OF FUNC FROM U=A TO U=B USING N INTERVALS.
 9005 H=(B-A)/FLOAT(N)
C     D IS ACCURACY OF J BESSEL FUNCTION DESIRED.
      D=.00001
C     CALCULATE BESSEL FUNCTIONS FOR LOWER LIMIT OF INTEGRAL, A=.001
      CALL BESJ(A,0,AJO,D,IER)
      AYOS=(AYO)**2
      AJOS=(AJO)**2
 9001 AR=A*R
      CALL BESJ(AR,0,AJOR,D,IER)
C     CALCULATE BESSEL FUNCTIONS OF UPPER LIMIT OF INTEGRAL, B.
      CALL BESJ(B,0,BJO,D,IER)
 9003 CALL BESY(B,0,BYO,IER)
      BJOS=(BJO)**2
      BYOS=(BYO)**2
      BR=B*R
      CALL BESJ(BR,0,BJOR,D,IER)
 9004 CALL BESY(BR,0,BYOR,IER)
      EVNSUM=0.0
      ODDSUM=0.0
      ENDSUM=FUNC(A,T,AJO,AYO,AJOR,AYOR,AJOS,AYOS,BETA)+FUNC(B,T,BJO,BYO
     2,BJOR,BYOR,BJOS,BYOS,BETA)
      M=N-1
      J=1
      DO 1500 I=1,M
      X=A+H*FLOAT(I)
C     CALCULATE BESSEL FUNCTIONS FOR X.
      CALL BESJ(X,0,XJO,D,IER)
      CALL BESY(X,0,XYO,IER)
      XJOS=(XJO)**2
      XYOS=(XYO)**2
      XR=X*R
      CALL BESJ(XR,0,XJOR,D,IER)
      CALL BESY(XR,0,XYOR,IER)
      GO TO (1001,1002),J
```

```
1001  ODDSUM=ODDSUM+FUNC(X,T,XJ0,XY0,XJ0R,XY0R,XJ0S,XY0S,BETA)
      J=2
      GO TO 1500
1002  EVNSUM=EVNSUM+FUNC(X,T,XJ0,XY0,XJ0R,XY0R,XJ0S,XY0S,BETA)
      J=1
1500  CONTINUE
1600  SIMP=(H/3.)*(ENDSUM+4.0*ODDSUM+2.0*EVNSUM)
      RETURN
      END
```

```
      COMPUTER PROGRAM 4.

      EXTERNAL FUNC
      DIMENSION AYORR(31)
C     A=LOWER LIMIT OF INTEGRAL
      A=.001
C     B=UPPER LIMIT OF INTEGRAL
      B=10.0
C     N=NUMBER OF INTERVALS
      N=290
C     BETA=BLOOD FLOW TERM
      BETA=.1
C     INPUT Y BESSEL VALUES FOR A AND AR
      AYORR(1)=-4.471416
      AYORR(2)=-4.410739
      AYORR(3)=-4.355347
      AYORR(4)=-4.304389
      AYORR(5)=-4.257210
      AYORR(6)=-4.213287
      AYORR(7)=-4.172200
      AYORR(8)=-4.133606
      AYORR(9)=-4.097217
      AYORR(10)=-4.062797
      AYORR(11)=-4.030141
      AYORR(12)=-3.99080
      AYORR(13)=-3.969463
      AYORR(14)=-3.941165
      AYORR(15)=-3.914070
      AYORR(16)=-3.888082
      AYORR(17)=-3.863112
      AYORR(18)=-3.839086
      AYORR(19)=-3.815933
      AYORR(20)=-3.793592
      AYORR(21)=-3.772010
      AYORR(22)=-3.751135
      AYORR(23)=-3.730922
      AYORR(24)=-3.711331
      AYORR(25)=-3.692327
      AYORR(26)=-3.673872
      AYORR(27)=-3.655937
      AYORR(28)=-3.638493
      AYORR(29)=-3.621515
      AYORR(30)=-3.604978
      AYORR(31)=-3.588861
      TSUM=0.0
      RSUM=0.0
START OUTER DO LOOP.  TIME VARIES FROM 0.1 TO 1.0
      DO 1000 I=1,10
```

```
C       START INNER DO LOOP.  RADIUS VARIES FROM 1.0 TO 4.0 FOR EACH
     VALUE OF TIME.
        DO 1100 K=1,31
        R=RSUM+1.0
        T=TSUM+0.1
        AYO=-4.4714155
        AYOR=AYORR(K)
C       EVALUATE INTEGRAL USING SIMPSON,S RULE.
        SINT=SIMP(FUNC,A,B,N,T,R,BETA,AYO,AYOR)
C       CALCULATE REMAINING TERMS IN THE EQUATION.
        W=SQRT(BETA)
        CALL BESK(W,0,BKW,IER)
        RKOB=BKW
        X=W*R
        CALL BESK(X,0,BKX,IER)
        RKOBR=BKX
C       COMBINE TERMS TO GET ANSWER.
        PI=3.14159265
        ANS=(RKOBR/RKOB)-(2/PI)*SINT*EXP(-BETA*T)
        WRITE(6,2000) T,R,ANS
 2000   FORMAT(' ','T=',F5.1,5X,'R=',F4.1,5X,'ANS=',F5.3)
C       RECALCULATE USING NEXT HIGHER VALUE OF R.
 1100   RSUM=RSUM+0.1
C       RECALCULATE FOR NEXT VALUE OF T.
 1000   TSUM=TSUM+0.1
        STOP
        END
C       FUNCTION FUNC SUBPROGRAM


        FUNCTION FUNC(U,T,RJOU,YOU,RJOUR,YOUR,RJOUS,YOUS,BETA)
        FUNC=EXP(-T*U**2)*((RJOU*YOUR)-(RJOUR*YOU))/((U+BETA/U)*(YOUS+RJOU
       2S))
        RETURN
        END
C       SIMPSON,S RULE SUBPROGRAM


        FUNCTION SIMP(FUNC,A,B,N,T,R,BETA,AYO,AYOR)
C       FUNC=FUNCTION NAME
C       A=INITIAL VALUE OF U
C       B=FINAL VALUE OF U
C       N=NUMBER OR INTERVALS,EVEN.
C       EVALUATE INTEGRAL OF FUNC FROM U=A TO U=B USING N INTERVALS.
 9005   H=(B-A)/FLOAT(N)
C       D IS ACCURACY OF J BESSEL FUNCTION DESIRED.
        D=.00001
C       CALCULATE BESSEL FUNCTIONS FOR LOWER LIMIT OF INTEGRAL, A=.001
```

```
      CALL BESJ(A,0,AJ0,D,IER)
      AYOS=(AY0)**2
      AJOS=(AJ0)**2
 9001 AR=A*R
      CALL BESJ(AR,0,AJOR,D,IER)
C     CALCULATE BESSEL FUNCTIONS OF UPPER LIMIT OF INTEGRAL, B.
      CALL BESJ(B,0,BJ0,D,IER)
 9003 CALL BESY(B,0,BY0,IER)
      BJOS=(BJ0)**2
      BYOS=(BY0)**2
      BR=B*R
      CALL BESJ(BR,0,BJOR,D,IER)
 9004 CALL BESY(BR,0,BYOR,IER)
      EVNSUM=0.0
      ODDSUM=0.0
      ENDSUM=FUNC(A,T,AJ0,AY0,AJOR,AYOR,AJOS,AYOS,BETA)+FUNC(B,T,BJ0,BY0
     2,BJOR,BYOR,BJOS,BYOS,BETA)
      M=N-1
      J=1
      DO 1500 I=1,M
      X=A+H*FLOAT(I)
C     CALCULATE BESSEL FUNCTIONS FOR X.
      CALL BESJ(X,0,XJ0,D,IER)
      CALL BESY(X,0,XY0,IER)
      XJOS=(XJ0)**2
      XYOS=(XY0)**2
      XR=X*R
      CALL BESJ(XR,0,XJOR,D,IER)
      CALL BESY(XR,0,XYOR,IER)
      GO TO (1001,1002),J
 1001 ODDSUM=ODDSUM+FUNC(X,T,XJ0,XY0,XJOR,XYOR,XJOS,XYOS,BETA)
      J=2
      GO TO 1500
 1002 EVNSUM=EVNSUM+FUNC(X,T,XJ0,XY0,XJOR,XYOR,XJOS,XYOS,BETA)
      J=1
 1500 CONTINUE
 1600 SIMP=(H/3.)*(ENDSUM+4.0*ODDSUM+2.0*EVNSUM)
      RETURN
      END
```

```
      COMPUTER PROGRAM 5.

      EXTERNAL FUNC
      DIMENSION AYORR(5)
C     A=LOWER LIMIT OF INTEGRAL
      A=.001
C     B=UPPER LIMIT OF INTEGRAL
      B=10.0
C     N=NUMBER OF INTERVALS
      N=290
C     BETA=BLOOD FLOW TERM
      BETA=.1
C     INPUT Y BESSEL VALUES FOR A AND AR
      DIMENSION AYORR(7)
      AYORR(1)=-3.588861
      AYORR(2)=-3.446794
      AYORR(3)=-3.330711
      AYORR(4)=-3.232564
      AYORR(5)=-3.147543
      AYORR(6)=-3.072545
      AYORR(7)=-3.005455
C     START OUTER DO LOOP.  TIME VARIES FROM 0.1 TO 1.0
      TSUM=0.0
      DO 1000 I=1,10
      RSUM=0.0
C     START INNER DO LOOP.  RADIUS VARIES FROM 4.0 TO10.0 FOR EACH
C     VALUE OF TIME.
      DO 1100 K=1,7
      R=RSUM+4.0
      T=TSUM+0.1
      AYO=-4.4714155
      AYOR=AYORR(K)
C     EVALUATE INTEGRAL USING SIMPSON,S RULE.
      SINT=SIMP(FUNC,A,B,N,T,R,BETA,AYO,AYOR)
C     CALULATE REMAINING TERMS IN THE EQUATION.
      W=SQRT(BETA)
      CALL BESK(W,0,BKW,IER)
      RKOB=BKW
      X=W*R
      CALL BESK(X,0,BKX,IER)
      RKOBR=BKX
C     COMBINE TERMS TO GET ANSWER.
      PI=3.14159265
      ANS=(RKOBR/RKOB)-(2/PI)*SINT*EXP(-BETA*T)
      WRITE(6,2000) T,R,ANS
 2000 FORMAT(' ','T=',F5.1,5X,'R=',F4.1,5X,'ANS=',F5.3)
C     RECALCULATE USING NEXT HIGHER VALUE OF R.
 1100 RSUM=RSUM+1.0
```

```
C       RECALCULATE FOR NEXT VALUE OF T.
 1000   TSUM=TSUM+0.1
        STOP
        END
C       FUNCTION FUNC SUBPROGRAM

        FUNCTION FUNC(U,T,RJOU,YOU,RJOUR,YOUR,RJOUS,YOUS,BETA)
        FUNC=EXP(-T*U**2)*((RJOU*YOUR)-(RJOUR*YOU))/((U+BETA/U)*(YOUS+RJOU
       2S))
        RETURN
        END
C       SIMPSON,S RULE SUBPROGRAM

        FUNCTION SIMP(FUNC,A,B,N,T,R,BETA,AYO,AYOR)
C       FUNC=FUNCTION NAME
C       A=INITIAL VALUE OF U
C       B=FINAL VALUE OF U
C       N=NUMBER OR INTERVALS,EVEN.
C       EVALUATE INTEGRAL OF FUNC FROM U=A TO U=B USING N INTERVALS.
 9005   H=(B-A)/FLOAT(N)
C       D IS ACCURACY OF J BESSEL FUNCTION DESIRED.
        D=.00001
C       CALCULATE BESSEL FUNCTIONS FOR LOWER LIMIT OF INTEGRAL, A=.001
        CALL BESJ(A,0,AJO,D,IER)
        AYOS=(AYO)**2
        AJOS=(AJO)**2
 9001   AR=A*R
        CALL BESJ(AR,0,AJOR,D,IER)
C       CALCULATE BESSEL FUNCTIONS OF UPPER LIMIT OF INTEGRAL, B.
        CALL BESJ(B,0,BJO,D,IER)
 9003   CALL BESY(B,0,BYO,IER)
        BJOS=(BJO)**2
        BYOS=(BYO)**2
        BR=B*R
        CALL BESJ(BR,0,BJOR,D,IER)
 9004   CALL BESY(BR,0,BYOR,IER)
        EVNSUM=0.0
        ODDSUM=0.0
        ENDSUM=FUNC(A,T,AJO,AYO,AJOR,AYOR,AJOS,AYOS,BETA)+FUNC(B,T,BJO,BYO
       2,BJOR,BYOR,BJOS,BYOS,BETA)
        M=N-1
        J=1
        DO 1500 I=1,M
        X=A+H*FLOAT(I)
C       CALCULATE BESSEL FUNCTIONS FOR X.
        CALL BESJ(X,0,XJO,D,IER)
```

```
      CALL BESY(X,0,XY0,IER)
      XJ0S=(XJ0)**2
      XY0S=(XY0)**2
      XR=X*R
      CALL BESJ(XR,0,XJ0R,D,IER)
      CALL BESY(XR,0,XY0R,IER)
      GO TO (1001,1002),J
1001  ODDSUM=ODDSUM+FUNC(X,T,XJ0,XY0,XJ0R,XY0R,XJ0S,XY0S,BETA)
      J=2
      GO TO 1500
1002  EVNSUM=EVNSUM+FUNC(X,T,XJ0,XY0,XJ0R,XY0R,XJ0S,XY0S,BETA)
      J=1
1500  CONTINUE
1600  SIMP=(H/3.)*(ENDSUM+4.0*ODDSUM+2.0*EVNSUM)
      RETURN
      END
```

COMPUTER PROGRAM 6.

```
      EXTERNAL FUNC
      DIMENSION AYORR(5)
C     A=LOWER LIMIT OF INTEGRAL
      A=.001
C     B=UPPER LIMIT OF INTEGRAL
      B=10.0
C     N=NUMBER OF INTERVALS
      N=290
C     BETA=BLOOD FLOW TERM
      BETA=.1
C     INPUT Y BESSEL VALUES FOR A AND AR
      AYORR(1)=-3.005456
      AYORR(2)=-2.563955
      AYORR(3)=-2.305486
      AYORR(4)=-2.121901
      AYORR(5)=-1.979311
C     START OUTER DO LOOP.  TIME VARIES FROM 0.1 TO 1.0
      TSUM=0.0
      DO 1000 I=1,10
      RSUM=0.0
C     START INNER DO LOOP.  RADIUS VARIES FROM 10.0 TO 50.0 FOR EACH
C     VALUE OF TIME.
      DO 1100 K=1,5
      R=RSUM+10.0
      T=TSUM+0.1
      AYO=-4.4714155
      AYOR=AYORR(K)
C     EVALUATE INTEGRAL USING SIMPSON,S RULE.
      SINT=SIMP(FUNC,A,B,N,T,R,BETA,AYO,AYOR)
C     CALULATE REMAINING TERMS IN THE EQUATION.
      W=SQRT(BETA)
      CALL BESK(W,0,BKW,IER)
      RK0B=BKW
      X=W*R
      CALL BESK(X,0,BKX,IER)
      RK0BR=BKX
C     COMBINE TERMS TO GET ANSWER.
      PI=3.14159265
      ANS=(RK0BR/RK0B)-(2/PI)*SINT*EXP(-BETA*T)
      WRITE(6,2000) T,R,ANS
 2000 FORMAT(' ','T=',F5.1,5X,'R=',F4.1,5X,'ANS=',F5.3)
C     RECALCULATE USING NEXT HIGHER VALUE OF R.
 1100 RSUM=RSUM+10.0
C     RECALCULATE FOR NEXT VALUE OF T.
 1000 TSUM=TSUM+0.1
      STOP
```

```
      END
C     FUNCTION FUNC SUBPROGRAM

      FUNCTION FUNC(U,T,RJOU,YOU,RJOUR,YOUR,RJOUS,YOUS,BETA)
      FUNC=EXP(-T*U**2)*((RJOU*YOUR)-(RJOUR*YOU))/((U+BETA/U)*(YOUS+RJOU
     2S))
      RETURN
      END
C     SIMPSON,S RULE SUBPROGRAM

      FUNCTION SIMP(FUNC,A,B,N,T,R,BETA,AYO,AYOR)
C     FUNC=FUNCTION NAME
C     A=INITIAL VALUE OF U
C     B=FINAL VALUE OF U
C     N=NUMBER OR INTERVALS,EVEN.
C     EVALUATE INTEGRAL OF FUNC FROM U=A TO U=B USING N INTERVALS.
 9005 H=(B-A)/FLOAT(N)
C     D IS ACCURACY OF J BESSEL FUNCTION DESIRED.
      D=.00001
C     CALCULATE BESSEL FUNCTIONS FOR LOWER LIMIT OF INTEGRAL, A=.001
      CALL BESJ(A,0,AJO,D,IER)
      AYOS=(AYO)**2
      AJOS=(AJO)**2
 9001 AR=A*R
      CALL BESJ(AR,0,AJOR,D,IER)
C     CALCULATE BESSEL FUNCTIONS OF UPPER LIMIT OF INTEGRAL, B.
      CALL BESJ(B,0,BJO,D,IER)
 9003 CALL BESY(B,0,BYO,IER)
      BJOS=(BJO)**2
      BYOS=(BYO)**2
      BR=B*R
      CALL BESJ(BR,0,BJOR,D,IER)
 9004 CALL BESY(BR,0,BYOR,IER)
      EVNSUM=0.0
      ODDSUM=0.0
      ENDSUM=FUNC(A,T,AJO,AYO,AJOR,AYOR,AJOS,AYOS,BETA)+FUNC(B,T,BJO,BYO
     2,BJOR,BYOR,BJOS,BYOS,BETA)
      M=N-1
      J=1
      DO 1500 I=1,M
      X=A+H*FLOAT(I)
C     CALCULATE BESSEL FUNCTIONS FOR X.
      CALL BESJ(X,0,XJO,D,IER)
      CALL BESY(X,0,XYO,IER)
      XJOS=(XJO)**2
      XYOS=(XYO)**2
```

```
      XR=X*R
      CALL BESJ(XR,0,XJOR,D,IER)
      CALL BESY(XR,0,XYOR,IER)
      GO TO (1001,1002),J
1001  ODDSUM=ODDSUM+FUNC(X,T,XJO,XYO,XJOR,XYOR,XJOS,XYOS,BETA)
      J=2
      GO TO 1500
1002  EVNSUM=EVNSUM+FUNC(X,T,XJO,XYO,XJOR,XYOR,XJOS,XYOS,BETA)
      J=1
1500  CONTINUE
1600  SIMP=(H/3.)*(ENDSUM+4.0*ODDSUM+2.0*EVNSUM)
      RETURN
      END
```

BIBLIOGRAPHY

1. Ambrosius, E.E., Fellows, R.D. and Brickman, A.D., Mechanical Measurement and Instrumentation, Ronald Press, 1966.

2. Aronow, S., "The Use of Radio-Frequency Power in Making Lesions in the Brain," J. Neurosurg., p. 431-438, May 1960.

3. Brodkey, J.S., Miyazaki, Y., Ervin, F.R. and Mark, V.H., "Reversible Heat with Radiofrequency Current," J. Neurosurg., v. 21, p. 49-53, 1964

4. Carpenter, M., and Whittier, J.R., "Study of Methods for Producing Experimental Lesions of the Central Nervous System with Special Reference to Stereotaxic Technique," J. of Comp. Neurology, v. 97, p. 73-117, 1952.

5. Carslaw, H.S., and Jaeger, J.C., Conduction of Heat in Solids, 2d ed., p. 327-338, Oxford Press, 1959.

6. Cooper, T.E., and Trezek, G.J., "Analytical Prediction of the Temperature Field Emanating From a Cryogenic Surgical Cannula," Cryobiology, v. 7, no. 2-3, p. 79-93.

7. Cooper, T.E., and Trezek, G.J., "Correlation of Thermal Properties of Some Human Tissue with Water Content", Aerospace Med., v. 42, p. 24-27, January 1971.

8. Daves, F., "Liquid Crystals: A new Tool for NDT," Research/ Development, p. 24-27, June 1967.

9. Dowden, W.A., "Cholesteric Liquid Crystals," Non-Destructive Testing, p. 99-102, November 1967.

10. Gengler, P.L., Heat Transfer Analysis of a Radio-frequency Brain Probe, M.S. Thesis, Naval Postgraduate School, Monterey, California, 1971.

11. Gildenberg, P.L., "Studies in Stereoencephalotomy X", Confinia Neurologica, v. 20, p. 53-65, 1960.

12. Gray, G.W., Molecular Structure and the Properties of Liquid Crystals, Academic Press, 1962.

13. Grayson, J., "Internal Calorimetry in the Determination of Thermal Conductivity and Blood Flow", J. Physiol., v. 118, p. 54-72, 1952.

14. Herrero, S., "Radio-Frequency-Current and Direct Current Lesions in the Ventromedial Hypothalamus," Am. J. Physsio., v. 217, no. 2, p. 403-410, August 1969.

15. Jaeger, J.C., "Numerical Values of the Temperature in Radial Heat Flow," J. Math. Phys., v. 34, p. 316-321, 1956.

16. Lawrence Radiation Lab, TRUMP: A Computer Program for Transient and Steady-State Temperature Distributions in Multi-dimensional Systems, Edwards, A.L., July, 1969.

17. National Cash Register Company, Encapsulated Liquid Crystals.

18. Naval Postgraduate School, M.E. Laboratory II Report, Liquid Crystals, by J.R. Fitts and others, December 1971.

19. Parker, R., Lawrence Livermore Laboratory, Personal conversation, 22 October 1971.

20. Pennes, H.H., "Analysis of Tissue and Arterial Blood Temperature in the Resting Human Forearm," J. Appl. Physiol., v. 1, no. 2, p. 93-122, August 1948.

21. Perl, W., "Heat and Matter Distribution in Body Tissue and the Determination of Tissue Blood Flow by Local Clearance Methods," J. Theoret. Biol., v. 2, p. 201-235, 1962.

22. Spells, K.E., "The Conductivities of Some Biological Fluids," Phys. Med. Biol. p. 139-1953, 1960.

23. Watkins , E.S., "Heat Gains in Brain During Electrocoagulative Lesions," J. Neurosurg., v. 23, p. 319-328, 1965.

24. Wilbur B. Driver Company, A Guide to Fine Wire Resistance Alloys for Electronic Applications.

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13. ABSTRACT

This thesis examines the use of heat for producing clinical lesions in tissue and presents the design and analysis of a resistively heated surgical probe. The probe temperature is accurately maintained and measured by using a Wheatstone bridge. The probe was embedded in an agar-water test medium and the temperature field emanating from the probe was measured using liquid crystal tapes. The experimental results compare within approximately 10% of a two-dimensional numerical solution. A one-dimensional theoretical model is developed and results are presented to show the relative effects of metabolism and blood flow on the temperature field.

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